Computer Aided Geometric Design

Analytical and relational properties

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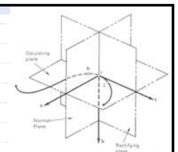
Analytic Properties

- Analytic properties generally fall into two categories:
  - Intrinsic (*local properties*)
    - **The intrinsic properties of a curve include**
    - the principal vectors (tangent, normal and binormal);
    - the principal planes (normal, osculating and rectifying);
    - curvature; and
    - the torsion.
    - **The intrinsic properties of a surface include**
    - the normal vector,
    - the tangent plane and
    - curvature.
  - Extrinsic (*global properties*)
    - For a curve, this includes its arc length and whether or not it is a plane curve or a straight line.
    - For a surface, global properties are such things as its area and whether or not it is ruled, developable, or planar.

Intrinsic properties of a curve



At any point on a curve, we can construct a set of three orthogonal vectors; together, called moving trihedron.

Tangent vector and line

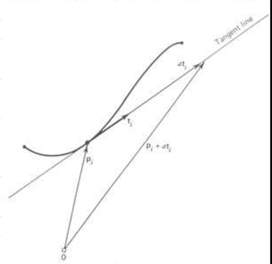
The tangent vector at point p_i on a curve is denoted by p_i^u.

$$t_i = \frac{p_i^u}{|p_i^u|}$$

The equation of a straight line through p_i and parallel to t_i is

$$q = p_i + at_i$$

Here, a is a scalar determining the distance along t_i from p_i.



Intrinsic properties of a curve

Normal Plane

The normal plane at point p_i on a curve is a plane through p_i perpendicular to the unit tangent vector t_i. The equation of the normal plane is given by the scalar product.

$$(q - p_i)\cdot t_i = 0$$

If q – p_i is perpendicular to ti, then the plane is also perpendicular to ti

Clearly, we can also write

$$(q - p_i)\cdot p_i^u = 0$$

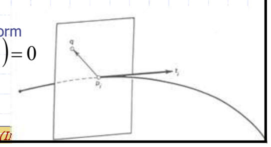
$$\begin{bmatrix} x - x_i \\ y - y_i \\ z - z_i \end{bmatrix}^T \begin{bmatrix} x_i^u \\ y_i^u \\ z_i^u \end{bmatrix} = 0$$

Now, denote the coordinates of the point q as x, y, z,

$$(x - x_i)x_i^u + (y - y_i)y_i^u + (z - z_i)z_i^u = 0$$

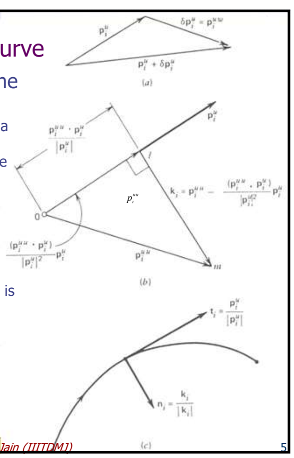
Equation for a plane, in ax + by+ cz + d=0, form

$$x_i^u x_i + y_i^u y_i + z_i^u z_i - (x_i x_i^u + y_i y_i^u + z_i z_i^u) = 0$$



Intrinsic properties of a curve

Principal normal vector and line



Principal normal vector at a point p_i on a curve lie on normal plane

It is a special normal vector towards the center of curvature for the point

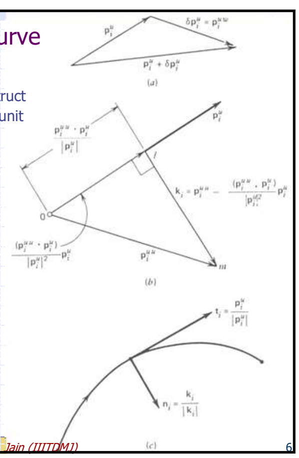
The tangent vector swings in the direction

$$p_i^{uu}$$

Length of the projection of p_i^{uu} onto p_i^u is

$$p_i^{uu} \cdot \left(\frac{p_i^u}{|p_i^u|} \right)$$

Intrinsic properties of a curve



Principal normal vector and line

Using projected length, a scalar, construct the vector along *0l* by multiplying the unit vector $p_i^u/|p_i^u|$ by the length found

$$\overline{0\,1} = \frac{p_i^{uu} \cdot p_i^u}{|p_i^u|^2} p_i^u$$

let k_i denote the vector from l to m. Applying simple vector arithmetic, we easily find that

$$k_i = p_i^{uu} - \frac{p_i^{uu} \cdot p_i^u}{|p_i^u|^2} p_i^u$$

Principal unit normal vector is

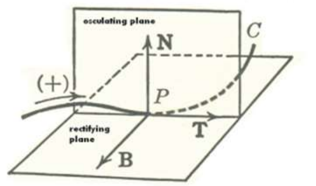
$$n_i = \frac{k_i}{|k_i|}$$

Intrinsic properties of a curve

Binormal vector

$$b_i = t_i \times n_i$$

It is also normal to n_i, so the three vectors form an orthogonal frame with considerable significance for geometric modeling.



Intrinsic properties of a curve

Osculating plane

The osculating plane at a point p_i on a curve is the limiting position of the plane defined by p_i and two neighboring points on the curve p_h and p_j as these neighbor points independently approach p_i .

These three points can not be colinear and the tangent vector p_i^u lies in this plane.

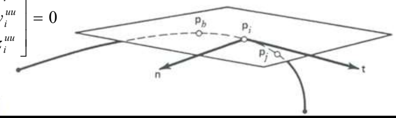
The equation of this plane is given by the following determinant:

$$\left[(q - p_i) \quad p_i^u \quad p_i^{uu}\right] = 0$$

Where q is any generic point on the osculating plane.

Writing the vectors in terms of their components, we obtain

$$\begin{bmatrix} x - x_i & x_i^u & x_i^{uu} \\ y - y_i & y_i^u & y_i^{uu} \\ z - z_i & z_i^u & z_i^{uu} \end{bmatrix} = 0$$



Intrinsic properties of a curve

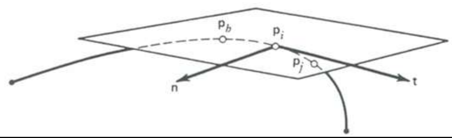
Osculating Plane

There is another way of formulating the osculating plane by using the vectors t and n, which we know from other considerations must lie in the plane.

we can formulate a parametric expression

$$q = p_i + ut_i + wn_i$$

Where q is any point on the osculating plane and u and w are parametric variables of unspecified bounds.



Intrinsic properties of a curve

Rectifying Plane

The rectifying plane at a point p_i on a curve is the plane through p_i and perpendicular to the principal normal n_i.

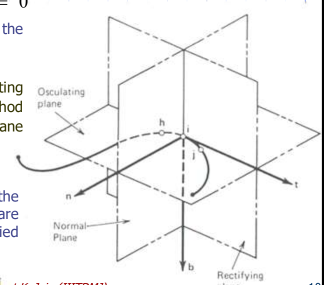
$$(r - p_i) \cdot n_i = 0$$

Where r is a generic point on the rectifying plane.

There is another way of formulating this plane, analogous to the method developed for the osculating plane and it is

$$r = p_i + ut_i + wb_i$$

where r is, any point on the rectifying plane and u and w are parametric variables of unspecified bounds



Intrinsic properties of a curve

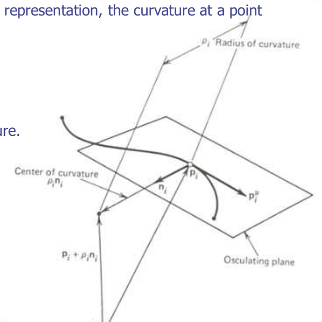
Curvature

For a space curve of parametric representation, the curvature at a point p_i on the curve is

$$\frac{1}{\rho_i} = \frac{\left|p_i^u \times p_i^{uu}\right|}{\left|p_i^u\right|^3}$$

ρ_i is called the radius of curvature. we have also used K to denote curvature where K=1/ ρ.

Curvature is measured in the osculating plane along the principal normal vector n_i. We can readily construct the curvature vector k= $\rho_i n_i$.

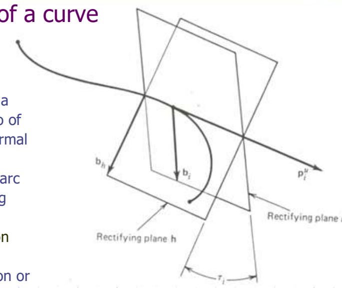


Intrinsic properties of a curve

Torsion

- The torsion at a point p_i on a curve is the limit of the ratio of the angle between the binormal at p_i and the binormal at a neighboring point p_h to the arc h_i, as p_h approaches p_i along the curve.
- It can be clearly observed on the rectifying planes.
- Torsion amounts to a rotation or twist about the tangent vector.
- It is given by the formula

$$\tau_i = \frac{\left|p_i^u \; p_i^{uu} \; p_i^{uuu}\right|}{\left|p_i^u \times p_i^{uu}\right|^2}$$



Intrinsic properties of a curve

Curvature and torsion are related

From the curvature equation $\frac{1}{\rho_i} = \frac{|p_i^u \times p_i^{uu}|}{|p_i^u|^3}$

So that $|p_i^u \times p_i^{uu}| = \frac{|p_i^u|^3}{\rho_i} =$

We can write $|p_i^u \times p_i^{uu}|^2 = \frac{|p_i^u|^6}{\rho_i^2}$

Substitute this in torsion equation $\tau_i = \frac{|p_i^u \; p_i^{uu} \; p_i^{uuu}|}{(1/\rho_i)|p_i^u|^6}$

Can be written as

$$\tau_i = \frac{|p_i^u \; p_i^{uu} \; p_i^{uuu}|}{|p_i^u|^6}\rho_i^2$$

Intrinsic properties of a curve

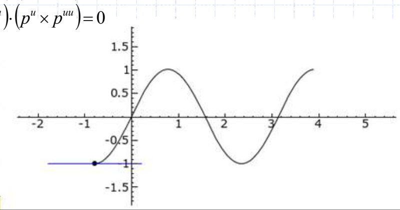
Inflection point

Points on a curve where the curvature equals zero and changes its sign are called inflection points

Inflection point can be find by solving $|p_i^u \times p_i^{uu}| = 0$

Any point on a curve with p_i^u parallel to p_i^{uu} is an inflection point.

$$|p^u \times p^{uu}| = \sqrt{(p^u \times p^{uu})\cdot(p^u \times p^{uu})} = 0$$
$$(p^u \times p^{uu})\cdot(p^u \times p^{uu}) = 0$$



Intrinsic properties of a Bezier curve

- Given a Bezier curve with control points
  - p_0 = [-2 -2 4]
  - p_1 = [2 -4 1]
  - p_2 = [6 -3 0]
  - p_3 = [10 0 0]
  - and p_4 = [10 4 2]
- Find the tangent vector and line, the normal plane, the principal normal vector and line, the binormal vector, the osculating plane, and rectifying plane at u=0.5
- Find the curvature, curvature vector, center of curvature, and torsion at u=0.5 for the Bezier curve defined above.
- Find any inflection point on the Bezier curve defined above.

First Fundamental forms

Let p= p(u, w) be a parametric surface patch;
then the first fundamental form is

Form I= $dp \cdot dp = Edu^2 + 2Fdudw + Gdw^2$

Where

$$E = p^u \cdot p^u$$
$$F = p^u \cdot p^w$$
$$G = p^w \cdot p^w$$

E, F, and G are known as the coefficients of the first fundamental form.
In the metric theory of surfaces, the first fundamental form arises in calculating the arc length of a curve on a surface.

Second Fundamental forms

Denote the unit normal to a surface at a point p(u, w) as n(u, w) or, more simply as n.
Then the second fundamental form is

Form II= $-dp \cdot dn = Ldu^2 + 2Mdudw + Ndw^2$

Where

$$L = -p^u \cdot n^u$$
$$M = -1/2\left(p^u \cdot n^w + p^w \cdot n^u\right)$$
$$N = -p^w \cdot n^w$$

L, M and N are known as the coefficients of the second fundamental form.
Note: n^u and n^w are perpendicular to n. Then, since p_u and p_w are perpendicular to n for all u, w, we can derive alternative expressions for L, M, and N i.e.

$$L = p^{uu} \cdot n$$
$$M = p^{uw} \cdot n$$
$$N = p^{ww} \cdot n$$

Intrinsic properties of a surface

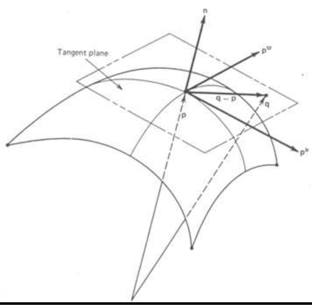
- Normal to a surface
- Tangent plane
- Principal curvature
- Geodesic and geodesic curvature
- Point Classification

Intrinsic properties of a surface

Normal to a surface

$$n = \frac{p^u \times p^w}{\left| p^u \times p^w \right|}$$

If p_u and p_w are perpendicular to each other, $p_u \bullet p_w = 0$, then p_u, p_w and n form a local orthogonal basis or trihedron at the point.



Intrinsic properties of a surface

Tangent plane

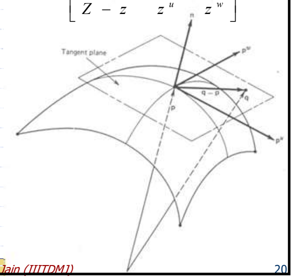
The equation of a plane tangent to a point on a surface is

$$\begin{vmatrix} X - x & x^u & x^w \\ Y - y & y^u & y^w \\ Z - z & z^u & z^w \end{vmatrix} = 0$$

Where (X, Y, Z) are the coordinates of any point q on the tangent plane; (x, y, z) are the coordinates of the point p on the surface to which the plane is tangent; and (x^u, y^u, z^u), (x^w, y^w, z^w) are components of the parametric tangent vectors p^u and p^w at p.

However, in what should be familiar vector terms, this determinant is

$$(q - p) \cdot \left(p^u \times p^w\right) = 0$$



Intrinsic properties of a surface

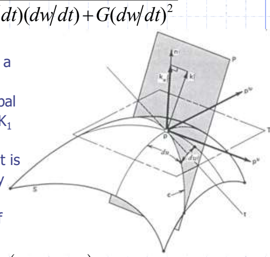
Principal curvature

$$\kappa_n = \frac{L(du/dt)^2 + 2M(du/dt)(dw/dt) + N(dw/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dw/dt) + G(dw/dt)^2}$$

When we rotate the plane around the normal, the curvature K_n varies and has a maximum and a minimum values in two perpendicular directions called as principal normal curvatures and denote them as K_1 and K_2.

A point on a surface where K_n= constant is called an *umbilical point*, and here every direction is a principal direction.

The principal curvatures are the roots of the following equation:



$$\left(EG - F^2\right)\kappa^2 - \left(EN + GL - 2FM\right)\kappa + \left(LN - M^2\right) = 0$$

Intrinsic properties of a surface

Find the principal directions as follows:

Let h= dw/du, then (FN - GM)h² + (EN-GL)h + EM – FL = 0

One of the roots of this equation makes the normal curvature a maximum, and the other makes it a minimum.

Gaussian and mean curvature

There are two other related measures of curvature for a surface. The first is called the Gaussian curvature K and is

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}$$

Which is an invariant property of the surface.

The second is called the mean curvature H and is

$$H = \frac{1}{2}\left(\kappa_1 + \kappa_2\right) = \frac{EN + GL - 2FM}{2\left(EG - F^2\right)}$$

Intrinsic properties of a surface

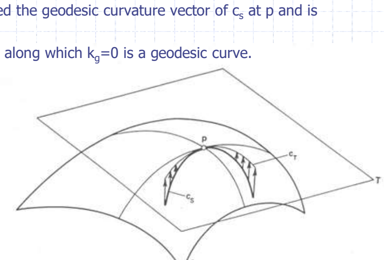
Geodesic and geodesic curvature

Given a curve c_s on a surface S,

at a point p on the curve and surface, we construct a tangent plane T.

The curve c_T is the orthogonal projection of c_s onto T. The curvature vector of c_T at p is called the geodesic curvature vector of c_s at p and is denoted by k_g.

Any curve on a surface along which k_g=0 is a geodesic curve.



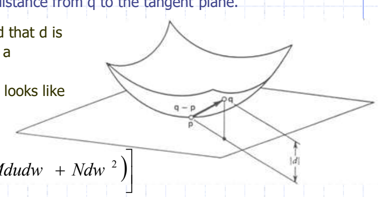
Intrinsic properties of a surface

Point Classification

Let d = (q – p)•n denote the projection of q-p onto n at p;

|d| is the perpendicular distance from q to the tangent plane.

Let q approach p, we find that d is increasingly governed by a function of the second fundamental form, which looks like



$$d = f\left[\frac{1}{2}\left(Ldu^2 + 2Mdudw + Ndw^2\right)\right]$$

This function is called the osculating paraboloid at p;

Its characteristics determine the nature of the surface in the neighborhood of the point.

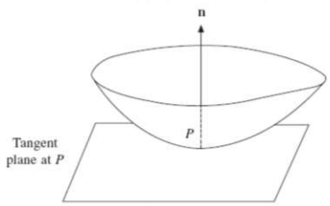
There are four important classes, depending on the value of $LN - M^2$

Intrinsic properties of a surface

Elliptical point

If $LN - M^2 > 0$, the point is an elliptic point.

In its neighborhood, the surface lies entirely on one side of the tangent plane, and the surface is locally convex with respect to the plane.

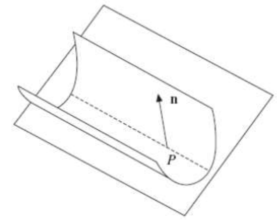


Intrinsic properties of a surface

Parabolic point

If $LN - M^2 = 0$ and $L^2 + M^2 + N^2 \neq 0$ the point is a parabolic point.

Here, we find a single line in the tangent plane along which d=0.



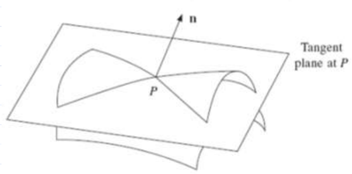
Intrinsic properties of a surface

Hyperbolic point

If $LN - M^2 < 0$, the point is a hyperbolic point.

Observe that there are two distinct line through p in the tangent plane dividing the plane into four regions.

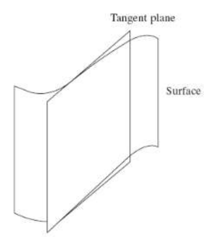
The surface lies on both sides of the tangent plane in the neighborhood of a hyperbolic point, and d is alternately positive and negative.



Intrinsic properties of a surface

Flat Point

If L=0, M=0, and N=0, then the point is a planar point.



Characteristic tests

- Planer curve ?
  - Curve should have zero torsion or twist

$$\left| p^u \quad p^{uu} \quad p^{uuu} \right| = 0 \qquad \begin{vmatrix} x^u & x^{uu} & x^{uuu} \\ y^u & y^{uu} & y^{uuu} \\ z^u & z^{uu} & z^{uuu} \end{vmatrix} = 0$$

- Straight line ?
  - Curvature must be zero at all points

$$\left| p^u \times p^{uu} \right| = 0$$

- Spherical ?
  - A surface is planar if $K_n=0$ at all points or spherical if K_n=constant
- Developable ?
  - A surface is developable if its Gaussian curvature K is zero

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} = 0$$

$$LN - M^2 = 0$$

Direct and inverse point solution

- Direct point Solution
  - Compute P(u) at predetermined increments of u
  - Forward difference method
- Inverse point solution
  - Finding the value of parametric variable corresponding to a given point

Global properties of curves and surfaces

- Depends on overall characteristics of geometric element
  - Arc length
  - Surface area
  - Volume

Global properties of curves and surfaces

Arc length

Compute length of the curve between u_1 and u_2

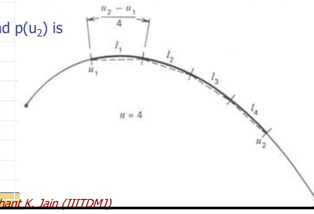
$$L = \sum_{i=1}^{n} l_i$$

Where $l_i = \sqrt{(p_i - p_{i-1})\cdot(p_i - p_{i-1})}$ and $p_i = \sum_{j=0}^{3} a_j u_i^j$

The arc length between $p(u_1)$ and $p(u_2)$ is

$$L = \int_{u_1}^{u_2} \sqrt{p^u \cdot p^u}\, du$$

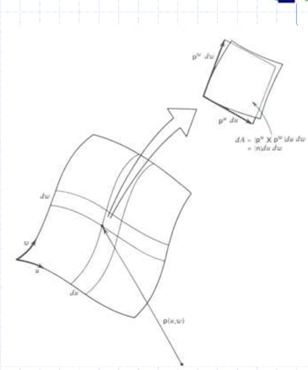
Where $u_2>u_1$



Global properties of curves and surfaces

Surface area

$$dA = |n|du\ dw$$
$$= f_1(u,w)du\ dw$$
$$A = \int_0^1\int_0^1 f_1(u,w)du\ dw$$



Global properties of curves and surfaces

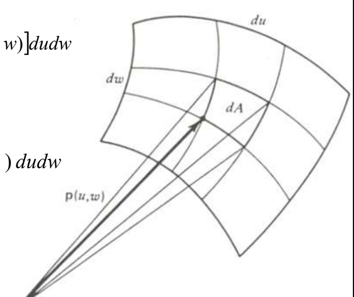
Volume

For the volume of a closed region, from vector calculus that

$$dV = \frac{1}{3}[p(u,w)\cdot n(u,w)]dudw$$
$$= f_2(u,w)dudw$$

We obtain

$$V = \int_0^1\int_0^1 f_2(u,w)\,dudw$$



Arc length of the Bezier curve

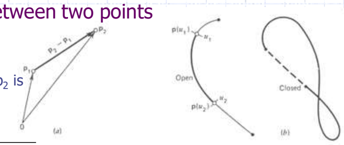
- Find the arc length of the Bezier curve whose control points are
  - p_0 = [10 0 0]
  - p_1 = [0 10 0]
  - p_2 = [0 -10 0]
  - p_3 = [10 -10 0]
- Find the arc length of the curve defined in previous question between u=0.4 and u=0.7.
- Find the arc length of the closed Bezier curve whose control points are
  - p_0 = [0 0 0]
  - p_1 = [10 0 0]
  - p_2 = [0 10 0]
  - p_3 = [-10 0 0]
  - p_4 = [0 0 0]
- Find the arc length of the curve defined in previous question between u=0.8 and u=0.1.

Relational Properties

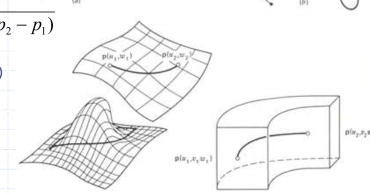
Minimum distance between two points

Minimum Distance between two points in space p_1 and p_2 is

$$d_{\min} = |p_2 - p_1|$$
$$= \sqrt{(p_2 - p_1)\cdot(p_2 - p_1)}$$



Between two points $p(u_1)$ and $p(u_2)$ on a curve



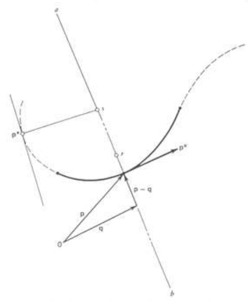
Minimum distance between two points on a surface or in a solid requires finding a common geodesic curve or curves passing through both points.

Minimum Distance between a Point and a Curve

$$d_{\min} = |p - q|$$

when

$$(p-q)\cdot p^u = 0$$

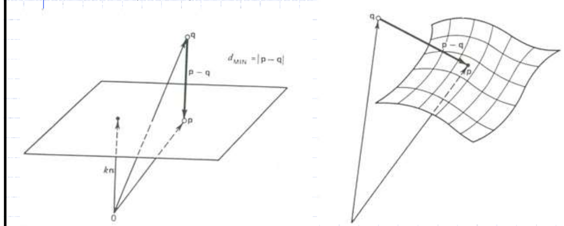


Minimum Distance between a Point and a Surface

$$d_{\min} = |p - q| \qquad (p-q)\times(p^u \times p^w) = 0$$

Where p must satisfy

$$(p - q)\times n = 0$$



Minimum Distance between Two Curves

$$d_{\min} = |p - q|$$

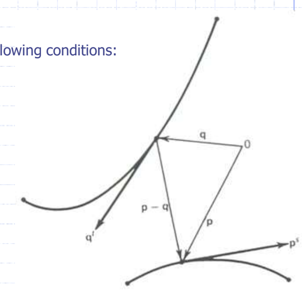
These points should satisfy the following conditions:

$$(p-q)\cdot p^s = 0$$

$$(p-q)\cdot q^t = 0$$

Two nonlinear simultaneous equations needs be solved.
Only those solution pairs (s_i, t_j) in the interval s, t ∈[0, 1] interest us.

Furthermore, we must also test the end points as well as for possible intersections.



Minimum Distance between a Curve and a Surface

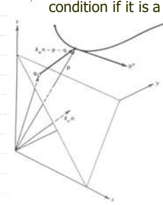
- The minimum distance is d_{min}= |p - q|, and these points satisfy the following conditions:

$$(p-q)\cdot p^u = 0$$

and

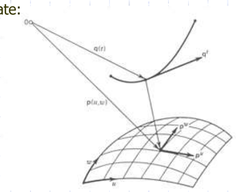
$$(p-q)\times n = 0$$

- For a bicubic surface patch the vector (p – q) must satisfy the following condition if it is a minimum-distance candidate:



$$(p-q)\times(p^u \times p^w) = 0$$

$$(p-q)\cdot q^t = 0$$



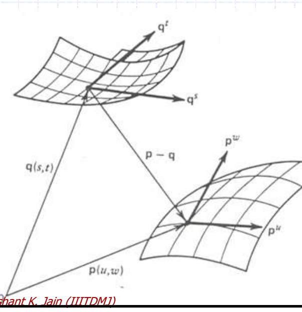
Minimum Distance between Two Surfaces

The minimum distance between two surface patches p(u, w) and q(s, t) is |p – q| when these vectors satisfy the following two conditions.

$$(p-q)\times(p^u \times p^w) = 0$$

and

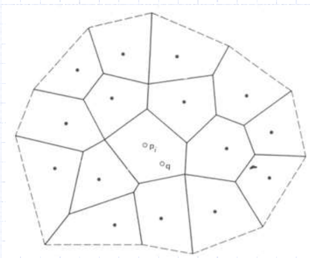
$$(p-q)\times(q^s \times q^t) = 0$$

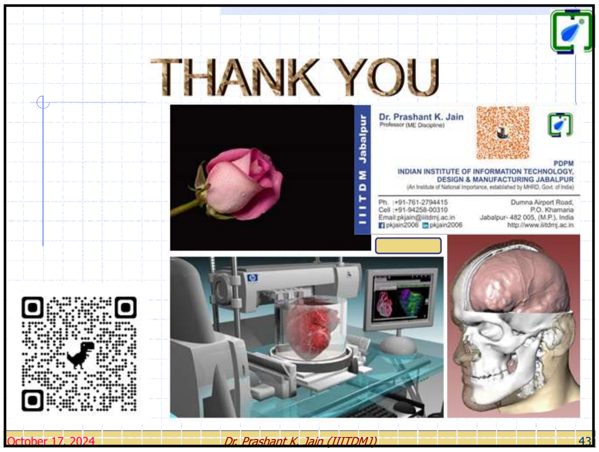


Nearest Neighbor spatial search

Given a set of points {p_i} in a plane and a point q not in this set, find the point in {p_i} closest to q.
This is classic problem in geometric sorting and searching.
One approach is to compute all d_i = |q – p_i| and then find the minimum.

The structure we choose is the Voronoi tesselation (also known as a Dirichlet tesselation).
Its characteristics are such that each point p_i is surrounded by a convex polygon.
All points within this polygon are closer to p_i than any other points in the set {p_i}.
This means that, given the Voronoi tesselation of a set of points {p_i} and point q, we determine the nearest neighboring point to it by finding the polygon in which q lies.

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October 17, 2024
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