Computer Aided Geometric Design

Transformations

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PDPM
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Transformations

- Translation
- Scaling
- Rotation
- Shear
- Reflection
- Matrix Notation
- Compositions/concatenation
- Homogeneous coordinates

Geometric Transformation

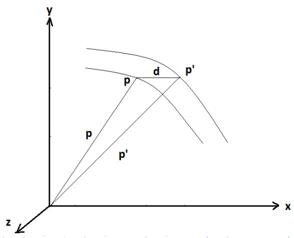
- Move points in space from one location to another
- Change the description of a point from one coordinate system to other like WCS to MCS
- Relative distance between object particles remain constant
- Referred as rigid body motion/transformation
- Include translation, scaling, reflection, rotation or combination

New position vector p' expressed in terms of the old position vector p and the motion parameters.

P' = f(p, transformation parameters)

P'= [T] P

[T] transformation matrix



Transformations

- **Assumption:** Objects consist of points and lines. A point is represented by its Cartesian coordinates:

  P=(x, y)

- **Geometric Transformation:**

  Let (A, B) be a straight line segment between the points A and B.

  Let T be a general 2D transformation.

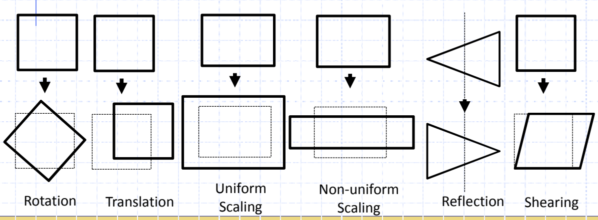
  T transforms (A, B) into another straight line segment (A', B'), where:

  A'=TA and

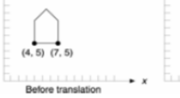
  B'=TB

Transformations

- Rotation
- Translation
- Scaling (uniform/non-uniform)
- Reflection
- Shear



Translation



- Rigid motion of points to new locations

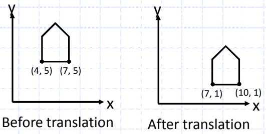
  $x' = x + d_x$

  $y' = y + d_y$

- Defined with column vectors:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

$$as \;\; P' = P + T$$

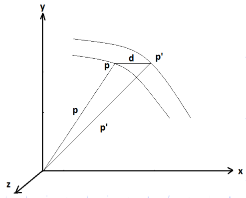


Translation

When every entity of a geometric model remains parallel to its initial position. It involves moving the elements from one location to another.

For a point

$$p' = p + d$$
$$x' = x + x_d$$
$$y' = y + y_d$$
$$z' = z + z_d$$
$$[x', y', z'] = [x, y, z][T]$$
$$T = [x_d, y_d, z_d]^T \quad \text{transformation matrix}$$



Scaling

- Used to change, increase or decrease the size of an entity or a model.
- Scaling is about the origin, that is a model changes size and location with respect to the origin. Model gets close or away from origin depending on scaling factor.

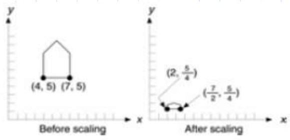
- Stretching of points along axes:

$$x' = s_x * x$$
$$y' = s_y * y$$

- In matrix form:

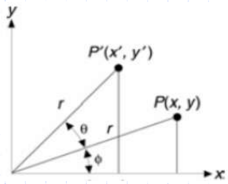
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$or \;\; just \quad P' = S \cdot P$$



Rotation

Enables users to view geometric models from different angles. Rotation has a unique characteristics that is not shared by translations, scaling, or reflection that is non-commutative ness. Two subsequent rotations of an entity about two different axes produce two different configurations for the entity depending on the order of the rotations.



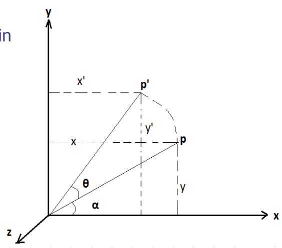
Rotation

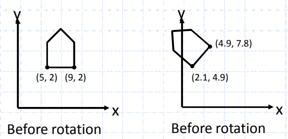
- Rotation of points about the origin

$$x = r \cdot \cos\phi$$
$$y = r \cdot \sin\phi$$
$$x' = r \cdot \cos(\theta + \phi)$$
$$= r \cdot \cos\phi \cdot \cos\theta - r \cdot \sin\phi \cdot \sin\theta$$
$$y' = r \cdot \sin(\theta + \phi)$$
$$= r \cdot \cos\phi . \sin\theta + r \cdot \sin\phi . \cos\theta$$
$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

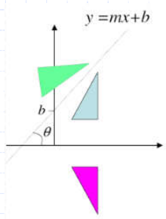
$$P' = R \cdot P$$





Reflection

Useful in constructing symmetric models, a geometric element can be reflected through a plane, a line or a point in space. Reflecting an entity through a principal plane is equivalent to negating the corresponding coordinate of each point on the entity, reflection through an axis is equivalent to reflection through two principal planes that intersect at the given axis.



Reflection

If an object P in xy plane is to be reflected about the x-axis i.e. y=0.

Reflection of a point (x, y) is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = Rfx \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

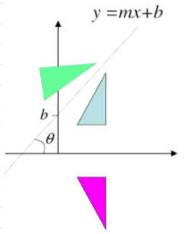
in the matrix form

$$P' = [M]P$$

$$M = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix} = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

Reflection

- Reflection is a special case of rotation when angle of rotation is 180° about an axis.



Shear

Produce distortion of an entity

will shear the point along the x-axis.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + Sh_x y \\ y \\ 1 \end{bmatrix}$$

will shear the point along the y-axis.

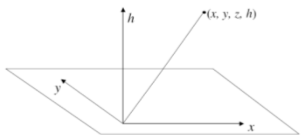
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y + Sh_y \\ 1 \end{bmatrix}$$

Homogeneous Representation

- Rotation, scaling, shearing, reflection are in form of matrix multiplication, but the translation takes the form of vector addition. This makes it inconvenient to concatenate transformation involving translation. It is desirable to express all geometric transformations in the form of matrix multiplication on representing points by their homogeneous coordinates.
- Provides an effective way to unify the description of geometric transformations as matrix multiplication.
- In homogeneous coordinates, an n-dimensional space is mapped into (n+1) dimensional space that is a point P(x,y,z) has homogeneous coordinates (x',y',z',h) where h is any scalar factor which is not equal to 0.

Homogeneous coordinates

- Homogeneous coordinates in 3 dimensions
- A point in homogeneous coordinates (x, y, z, h), h≠0, corresponds to the 3-D vertex (x/h, y/h, z/h) in Cartesian coordinates.
- Homogeneous coordinates in 3D give rise to 4 dimensional position vector



Homogeneous coordinates

$$x = \frac{x'}{h}, y = \frac{y'}{h}, z = \frac{z'}{h}$$

Translation matrices can now be written as:

$$[x' \quad y' \quad z' \quad 1]^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_d & y_d & z_d & 1 \end{bmatrix}^T * [x \quad y \quad z \quad 1]^T$$

Homogeneous coordinates

- Homogeneous coordinates for 2D space requires 3D vectors and matrices.
- Homogeneous coordinates for 3D space requires 4D vectors and matrices.

$$S(S_x, S_y, S_z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Matrix Representation for 2D Transformation

- Translation: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- Scale: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- Rotation: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- Shear: $SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Reflection: $F_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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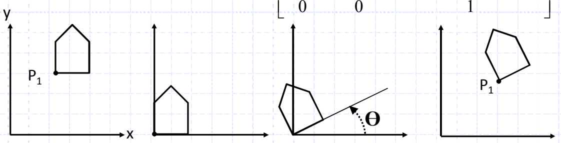
Rotation about an arbitrary point

- Rotate about a point P
  - Translate P1 to origin
  - Rotate
  - Translate back to P1

$$T(x_1, y_1) \cdot R(\theta) \cdot T(-x_1, -y_1)$$

$$= \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_1(1-\cos\theta) + y_1 \sin\theta \\ \sin\theta & \cos\theta & y_1(1-\cos\theta) - x_1 \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$



Original house | After translation of P_1 to origin | After rotation | After translation to original P_1

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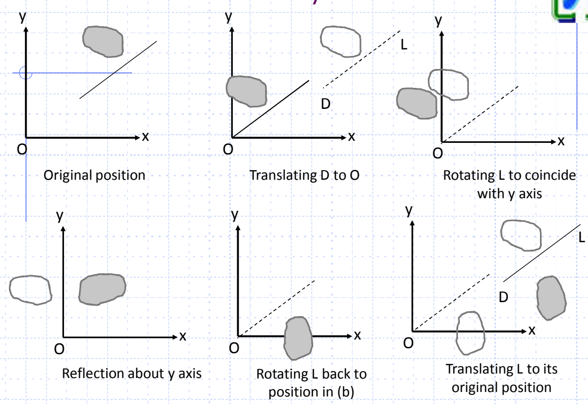
Reflection about an arbitrary line



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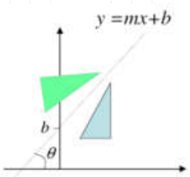
Reflection about an arbitrary line

- Translate (0, -b) so that the line passes through the origin.
- Rotate the line about the z axis by -θ°.
- Reflect object about the x axis.
- Rotate back the line by θ°.
- Translate back (0, b).



$$[T] = [T_t][T_r][R][T_r]^{-1}[T_t]^{-1}$$

$$\theta = \tan^{-1}(m)$$

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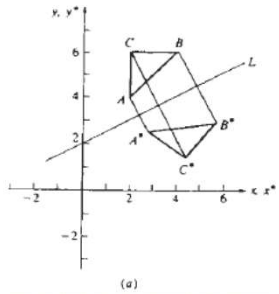
Reflect triangle (2,4), (4,6), (2,6) about line

y= ½ (x+4)



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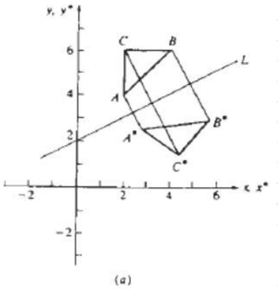
- Consider the line L and the triangle ABC shown in Figure. The equation of the line L is

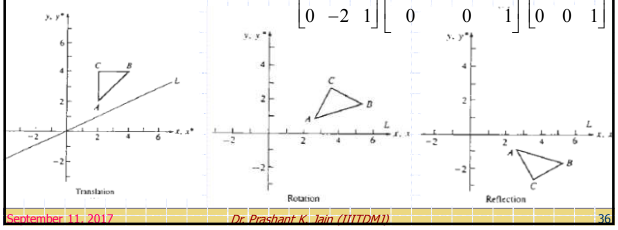
$$y=\frac{1}{2}(x+4)$$

- The position vectors [2 4 1], [4 6 1] and [2 6 1] describe the vertices of the triangle ABC in homogeneous coordinate.



- The line L will pass through the origin by transiting it -2 units in the y direction.
- The resulting line can be made coincident with the x-axis by rotating it, by $-\tan^{-1}(2/4) = -26.57$ about the origin (CW).
- Then reflect the triangle through the x-axis.

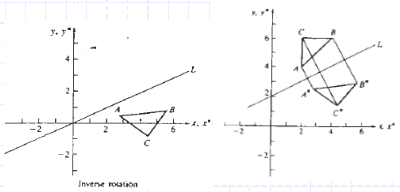
$$[T]=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\cdot\begin{bmatrix}2/\sqrt{5}&-1/\sqrt{5}&0\\1/\sqrt{5}&2/\sqrt{5}&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$



- The transformed position vectors of the triangle are then rotated (by 26.57) and translated (by 2 units in the y direction) back to the original orientation.
- The combined transformation is:

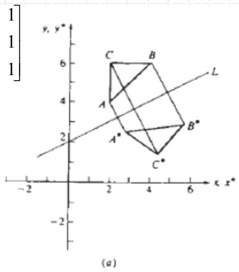
$$[T]=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\cdot\begin{bmatrix}2/\sqrt{5}&-1/\sqrt{5}&0\\1/\sqrt{5}&2/\sqrt{5}&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}2/\sqrt{5}&1/\sqrt{5}&0\\-1/\sqrt{5}&2/\sqrt{5}&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}$$

$$[T]=\begin{bmatrix}3/5&4/5&0\\4/5&-3/5&0\\-8/5&16/5&1\end{bmatrix}$$

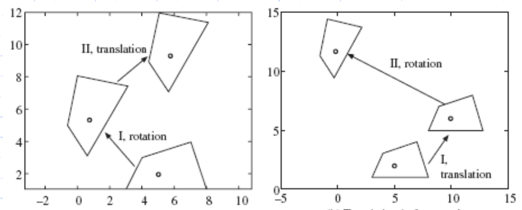


The transformed position for the triangle $A^*B^*C^*$ are

$$\begin{bmatrix}2&4&1\\4&6&1\\2&6&1\end{bmatrix}\begin{bmatrix}3/5&4/5&0\\4/5&-3/5&0\\-8/5&16/5&1\end{bmatrix}=\begin{bmatrix}14/5&12/5&1\\28/5&14/5&1\\22/5&6/5&1\end{bmatrix}$$



Significance of order in transformation



(a) Rotation before translation (b) Translation before rotation

3D Translation and Scaling

- Translation
- Scale
  - Parameters for each axis direction

$$T(s_x,s_y,s_z)=\begin{bmatrix}s_x&0&0&0\\0&s_y&0&0\\0&0&s_z&0\\0&0&0&1\end{bmatrix}$$

3D Rotation

- One rotation for each world coordinate axis

$$R_x(\theta)=\begin{bmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{bmatrix}$$

$$R_y(\theta)=\begin{bmatrix}\cos\theta&0&\sin\theta&0\\0&1&0&0\\-\sin\theta&0&\cos\theta&0\\0&0&0&1\end{bmatrix}$$

$$R_z(\theta)=\begin{bmatrix}\cos\theta&-\sin\theta&0&0\\\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

3D Shearing

- Shearing:

$$\begin{bmatrix}1&a&b&0\\c&1&d&0\\e&f&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}x+ay+bz\\cx+y+dz\\ex+fy+z\\1\end{bmatrix}$$

- Change in each coordinate is a linear combination of all three.
- Transforms a cube into a general parallelepiped.

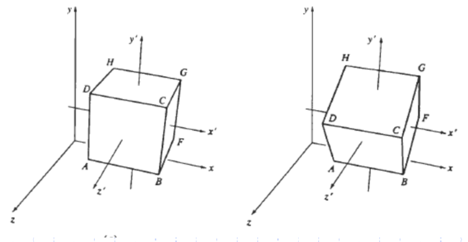
3D Reflection

- A reflection through the xy plane:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}x\\y\\-z\\1\end{bmatrix}$$

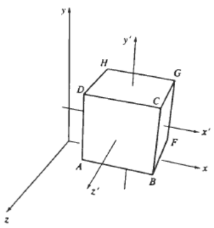
- Reflections through the xz and the yz planes are defined similarly.

Rotation about an axis parallel to a coordinate axis



- Consider the block shown in figure by the position vectors relative to the global *xyz*-axis system.

$$[X]=\begin{bmatrix}1&1&2&1\\2&1&2&1\\2&2&2&1\\1&2&2&1\\1&1&1&1\\2&1&1&1\\2&2&1&1\\1&2&1&1\end{bmatrix}\begin{matrix}A\\B\\C\\D\\E\\F\\G\\H\end{matrix}$$



- Let's rotate the block by $\vartheta = +30°$ about the local *x*-axis passing through the centroid of the block. The origin of the local axis system is assumed to be the centroid of the block

- The centroid of the block is $[x_c\ y_c\ z_c\ 1]$ = [3/2 3/2 3/2 1].
- The rotation is accomplished by

$$[X^\circ]=[X]\cdot[Tr]\cdot[R_z]\cdot[Tr]^{-1}$$

- where

$$[Tr]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-y_c&-z_c&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-3/2&-3/2&1\end{bmatrix}$$

$$[R_x]=\begin{bmatrix}1&0&0&0\\0&\cos\theta&\sin\theta&0\\0&-\sin\theta&\cos\theta&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&0.866&0.5&0\\0&-0.5&0.866&0\\0&0&0&1\end{bmatrix}$$

- and

$$[Tr]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_c & z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3/2 & 3/2 & 1 \end{bmatrix}$$

- The first matrix [*Tr*] translates the block parallel to the *x*=0 plane until the *x'*-axis is coincident with that *x*-axis.
- The second matrix [*Rz*] performs the required rotation about the *x*-axis,
- and the third matrix $[Tr]^{-1}$ translates the *x'*-axis and hence the rotated block back to its original position.

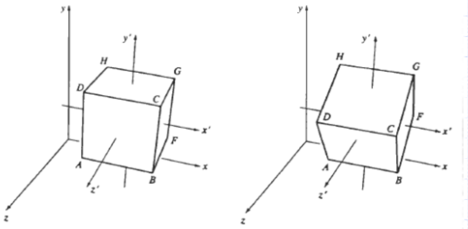
- Concatenating these three matrices yields.

$$[T]=[Tr]\cdot[R_z]\cdot[Tr]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & y_c(1-\cos\theta)+z_c\sin\theta & z_c(1-\cos\theta)-y_c\sin\theta & 1 \end{bmatrix}$$

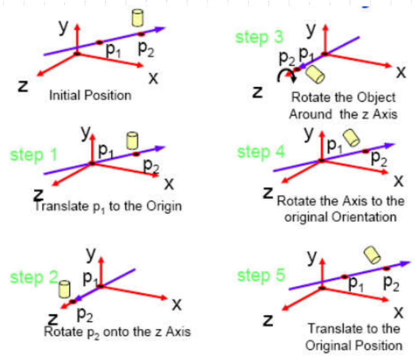
- After subtracting numerical values the transformed coordinates are

$$[X']=[X]\cdot[T]= \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.951 & -0.549 & 1 \end{bmatrix}$$

$$[X] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{matrix} \qquad [X'] = \begin{bmatrix} 1 & 0.817 & 1.683 & 1 \\ 2 & 0.817 & 1.683 & 1 \\ 2 & 1.683 & 2.183 & 1 \\ 1 & 1.683 & 2.183 & 1 \\ 1 & 1.317 & 0.817 & 1 \\ 2 & 1.317 & 0.817 & 1 \\ 2 & 2.183 & 1.317 & 1 \\ 1 & 2.183 & 1.317 & 1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{matrix}$$

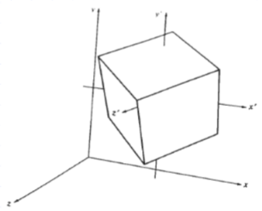


Rotation about an arbitrary axis in 3D



Rotation About an Arbitrary Axis in Space

- Translate so that the point (x_0, y_0, z_0) is at the origin of the coordinate system.
- Perform approximate rotation to make the axis of rotation with the *z*-axis.
- Rotate about the *z*-axis by the angle *δ*.
- Perform the inverse of the combined rotation transformation.
- Perform the inverse of the translation.



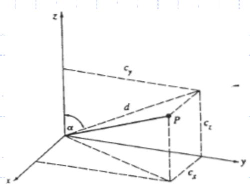
In general, making an arbitrary axis passing the origin coincide with one of the coordinate axes requires *two successive rotations about the other two coordinate axes.*

To make the arbitrary rotation axis coincident with the z-axis, first rotate about the x-axis and then about the y-axis.

To determine the rotation angle α about the x-axis used to place the arbitrary axis in the xz plane, first project the unit vector along the x-axis onto the yz plane. The y and z components of the projected vector are c_y and c_z, the direction cosine of the unit vector along the arbitrary axis. As shown in figure we have that

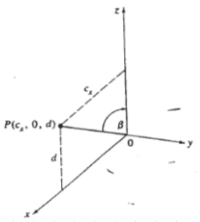
$$d = \sqrt{c_y^{\,2} + c_z^{\,2}}$$

$$\cos\alpha = \frac{c_z}{d}, \quad \sin\alpha = \frac{c_y}{d}$$



After rotation about the x-axis into the xz plane, the z component of the unit vector is *d*, and the x component is c_x, the direction cosine in the x direction. Thus the rotation angle β about the y-axis required to make the arbitrary axis coincident with z-axis is

$\cos\beta = d \qquad \sin\beta = c_x$



Rotation About an Arbitrary Axis in Space

The complete transformation is then

$$[M]=[T]\cdot[R_x]\cdot[R_y]\cdot[R_\delta][R_y]^{-1}[R_x]^{-1}[T]^{-1}$$

where the required translation matrix is

$$[T]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-x_0&-y_0&-z_0&1\end{bmatrix}$$

The transformation matrix for rotation about the *x*-axis is

$$[R_x]=\begin{bmatrix}1&0&0&0\\0&\cos\theta&\sin\theta&0\\0&-\sin\theta&\cos\theta&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&c_z/d&c_y/d&0\\0&-c_y/d&c_z/d&0\\0&0&0&1\end{bmatrix}$$

And about the *y*-axis

$$[R_y]=\begin{bmatrix}\cos(-\beta)&0&\sin(-\beta)&0\\0&1&0&0\\-\sin(-\beta)&0&\cos(-\beta)&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}d&0&c_x&0\\0&1&0&0\\-c_x&0&d&0\\0&0&0&1\end{bmatrix}$$

Finally, the rotation about the arbitrary axis is given by a *z*-axis rotation matrix

$$[R_\delta]=\begin{bmatrix}\cos\delta&\sin\delta&0&0\\-\sin\delta&\cos\delta&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

If the direction of the arbitrary axes are not known then they can be obtained knowing a second point on the axis (x_1, y_1, z_1) by normalizing the vector from the first to the second point. Specifically, the vector along the axis from (x_0, y_0, z_0) to (x_1, y_1, z_1) is

$$[V] = [(x_1 - x_0)\ (y_1 - y_0)\ (z_1 - z_0)]$$

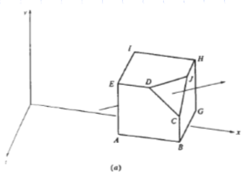
Normalized, it yields

$$[c_x\ \ c_y\ \ c_z]=\frac{[(x_1-x_0)\ \ (y_1-y_0)\ \ (z_1-z_0)]}{[(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2]^{1/2}}$$

Consider the cube with one corner removed

vectors for the vertices are

$$[X]=\begin{bmatrix}2&1&2&1\\3&1&2&1\\3&1.5&2&1\\2.5&2&2&1\\2&2&2&1\\2&1&1&1\\3&1&1&1\\3&2&1&1\\2&2&1&1\\3&2&1.5&1\end{bmatrix}\begin{matrix}A\\B\\C\\D\\E\\F\\G\\H\\I\\J\end{matrix}$$



The cube is to be rotated by -45° about a local axis passing through the point *F* and the diagonally opposite corner. The axis is directed from F to the opposite corner and passes through the center of the corner face.

Determine the direction cosine of the rotation axis. Observing that the corner cut off by the triangle *CDJ* also lies on the axis of rotation.

$$[c_x\ \ c_y\ \ c_z]=\frac{[(3-2)\ \ (2-1)\ \ (2-1)]}{[(3-2)^2+(2-1)^2+(2-1^2)]^{1/2}}$$

$$=\left[\tfrac{1}{\sqrt{3}}\ \ \tfrac{1}{\sqrt{3}}\ \ \tfrac{1}{\sqrt{3}}\right]$$

Using Equations

$$d=\sqrt{c_y^{\,2}+c_z^{\,2}} \qquad \cos\alpha=\frac{c_z}{d},\ \sin\alpha=\frac{c_y}{d} \qquad \cos\beta=d,\ \sin\beta=c_x$$

yields

$$d=\sqrt{\left(\tfrac{1}{\sqrt{3}}\right)^2+\left(\tfrac{1}{\sqrt{3}}\right)^2}=\sqrt{\tfrac{2}{3}} \qquad \alpha=\cos^{-1}\left(\frac{1/\sqrt{3}}{\sqrt{2/3}}\right)=\cos^{-1}\left(\tfrac{1}{\sqrt{2}}\right)=45^0$$

$$\beta=\cos^{-1}\left(\tfrac{2}{3}\right)=35.26^0$$

Since the point *F* lies on the rotation axis, the translation matrix and the rotation matrices to make the arbitrary axis coincident with the *z*-axis are

$$[T]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-2&-1&-1&1\end{bmatrix}\quad [R_x]=\begin{bmatrix}1&0&0&0\\0&1/\sqrt{2}&1/\sqrt{2}&0\\0&-1/\sqrt{2}&1/\sqrt{2}&0\\0&0&0&1\end{bmatrix}\quad [R_y]=\begin{bmatrix}2/\sqrt{6}&0&1/\sqrt{3}&0\\0&1&0&0\\-1/\sqrt{3}&0&2/\sqrt{6}&0\\0&0&0&1\end{bmatrix}$$

Concatenating $[T][R_x][R_y]$ yields

$$[M]=[T]\cdot[R_x]\cdot[R_y]=\begin{bmatrix}2/\sqrt{6}&0&1/\sqrt{3}&0\\-1/\sqrt{6}&1/\sqrt{2}&1/\sqrt{3}&0\\-1/\sqrt{6}&-1/\sqrt{2}&1/\sqrt{3}&0\\-2/\sqrt{6}&0&-4/\sqrt{3}&1\end{bmatrix}$$

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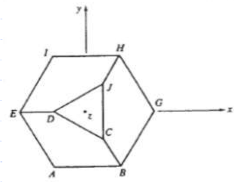
The transformed intermediate position vectors are

$$[X]\cdot[M]=\begin{bmatrix}-0.408&-0.707&0.577&1\\0.408&-0.707&1.155&1\\0.204&-0.354&1.443&1\\-0.408&0&1.443&1\\-0.816&0&1.155&1\\0&0&0&1\\0.816&0&0.577&1\\0.408&0.707&1.155&1\\-0.408&0.707&0.577&1\\0.204&0.354&1.443&1\end{bmatrix}$$



Notice that point *F* is at (0, 0, 0)

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The rotation about the arbitrary axis is now given by the equivalent rotation about the *z*-axis.

$$[R_\delta]=\begin{bmatrix}\sqrt{2}/2&-\sqrt{2}/2&0&0\\\sqrt{2}/2&\sqrt{2}/2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

The transformed object is returned to its 'original' position in space, using

$$[M]^{-1}=[R_y]^{-1}\cdot[R_x]^{-1}\cdot[T]^{-1}=\begin{bmatrix}2/\sqrt{6}&-1/\sqrt{6}&-1/\sqrt{6}&0\\0&1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{3}&1/\sqrt{3}&1/\sqrt{3}&0\\2&1&1&1\end{bmatrix}$$

This result can be obtained by concatenating the inverse of the individual component matrices of [M] or by formally taking the inverse of [*M*]. $[R_x]^{-1}$, $[R_y]^{-1}$, and $[T]^{-1}$ are obtained by substituting $-\alpha$, $-\beta$ and (x_0, y_0, z_0) for α, β and $(-x_0, y_0, z_0)$, respectively.

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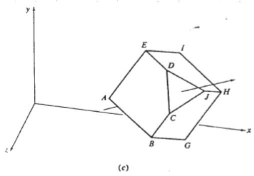
The resulting position vectors are

$$[X]\cdot[M]\cdot[R_\delta]\cdot[M]^{-1}=\begin{bmatrix}1.689&1.506&1.805&1\\2.494&1.195&2.311&1\\2.747&1.598&2.155&1\\2.598&2.155&1.747&1\\2.195&2.311&1.494&1\\2&1&1&1\\2.805&0.689&1.506&1\\3.311&1.494&1.195&1\\2.506&1.805&0.689&1\\3.155&1.747&1.598&1\end{bmatrix}$$



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Reflection Through an Arbitrary Plane

- Translate a known point *P*, that lies in the reflection plane, to the origin of the coordinate system.
- Rotate the normal vector to the reflection plane at the origin until it is coincident with the +*z*-axis; this makes the reflection plane the *z*=0 coordinate plane.
- After applying the above transformations to the object, reflect the object through the *z*=0 coordinate plane
- Perform the inverse transformations to those given above to achieve the desired result.

The general transformation is then:

$$[M]=[T]\cdot[R_x]\cdot[R_y]\cdot[Rflt_z]\cdot[R_y]^{-1}\cdot[R_x]^{-1}\cdot[T]^{-1}$$

Where the matrices [*T*], $[R_x]$, $[R_y]$ are obtained for translation and rotations, respectively. $(x_0, y_0, z_0) = (P_x, P_y, P_z)$, the components of point *P* in the reflection plane; and (c_x, c_y, c_z) are the direction cosines of the normal to the reflection plane.

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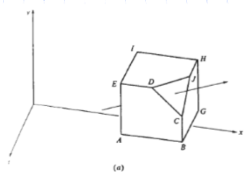
Consider the cube with one corner removed vectors for the vertices are

$$[X]=\begin{bmatrix}2&1&2&1\\3&1&2&1\\3&1.5&2&1\\2.5&2&2&1\\2&2&2&1\\2&1&1&1\\3&1&1&1\\3&2&1&1\\2&2&1&1\\3&2&1.5&1\end{bmatrix}\begin{matrix}A\\B\\C\\D\\E\\F\\G\\H\\I\\J\end{matrix}$$



The cube is to be reflected through the plane containing the triangle CDJ.

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The normal to the reflection plane is obtained using the position vectors C, D, J.

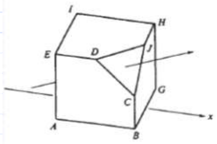
Specifically, taking the cross product of the vectors CJ and CD prior to translation yields

n = ([J] – [C]) X ([D] – [C])

= [(3 - 3) (2 – 1.5) (1.5 - 2)] X [(2.5 – 3) (2 – 1.5) (2 – 2)]

= [0 1/2 -1/2] X [-1/2 1/2 0]

= [1/4 1/4 1/4]



Normalizing yields

$$= \left[\tfrac{1}{\sqrt{3}} \;\; \tfrac{1}{\sqrt{3}} \;\; \tfrac{1}{\sqrt{3}} \right]$$

Using Equations

$$d = \sqrt{c_y^{\,2} + c_z^{\,2}} \qquad \cos\alpha = \frac{c_z}{d}, \;\; \sin\alpha = \frac{c_y}{d} \qquad \cos\beta = d, \;\; \sin\beta = c_x$$

yields

$$d = \sqrt{\left(\tfrac{1}{\sqrt{3}}\right)^2 + \left(\tfrac{1}{\sqrt{3}}\right)^2} = \sqrt{\tfrac{2}{3}} \qquad \alpha = \cos^{-1}\left(\frac{1/\sqrt{3}}{\sqrt{2/3}}\right) = \cos^{-1}\left(\tfrac{1}{\sqrt{2}}\right) = 45^0$$

$$\beta = \cos^{-1}\left(\tfrac{2}{3}\right) = 35.26^0$$

Translate the point *C* to the origin, yield the translation matrix

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -3/2 & -2 & 1 \end{bmatrix}$$

The rotation matrices to make the normal at *C* coincide with *z*-axis are

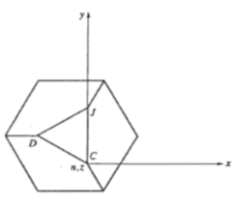
$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [R_y] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenating [T], $[R_x]$ and $[R_y]$ yields

$$[M] = [T]\cdot[R_x]\cdot[R_y] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -5/2\sqrt{6} & 1/2\sqrt{2} & -13/2\sqrt{3} & 1 \end{bmatrix}$$

- The transformed intermediate position vectors are

$$[X]\cdot[M] = \begin{bmatrix} -0.612 & -0.354 & -0.876 & 1 \\ 0.204 & -0.354 & -0.287 & 1 \\ 0 & 0 & 0 & 1 \\ -0.612 & 0.354 & 0 & 1 \\ -1.021 & 0.354 & -0.287 & 1 \\ -0.204 & 0.354 & -1.443 & 1 \\ 0.612 & 0.354 & -0.876 & 1 \\ 0.204 & 1.061 & -0.287 & 1 \\ -0.612 & 1.061 & -0.876 & 1 \\ 0 & 0.707 & 0 & 1 \end{bmatrix}$$



Notice that the point *C* is at the origin and the *z*-axis points out of the screen.

Reflection through the arbitrary plane is now given by reflection through the *z*=0 plane.

$$[Rflt] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

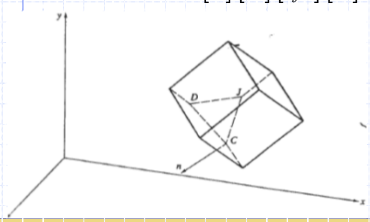
Returning the transformed object to its 'original' position in space requires

$$[M]^{-1} = [R_y]^{-1}\cdot[R_x]^{-1}\cdot[T]^{-1} = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 3 & 3/2 & 2 & 1 \end{bmatrix}$$

The matrices $[R_x]^{-1}$, $[R_y]^{-1}$ and $[T]^{-1}$ are obtained by substituting –α, -β, and $[x_0 \;\; y_0 \;\; z_0]$ = C.

The resulting position vectors are

$$[X]\cdot[M]\cdot[Rflt]\cdot[M]^{-1} = \begin{bmatrix} 3 & 2 & 3 & 1 \\ 10/3 & 4/3 & 7/3 & 1 \\ 3 & 3/2 & 2 & 1 \\ 5/2 & 2 & 2 & 1 \\ 7/3 & 7/3 & 7/3 & 1 \\ 11/3 & 8/3 & 8/3 & 1 \\ 4 & 2 & 2 & 1 \\ 10/3 & 7/3 & 4/3 & 1 \\ 3 & 3 & 2 & 1 \\ 3 & 2 & 3/2 & 1 \end{bmatrix}$$



- Generalized 4X4 transformation matrix in homogeneous coordinates

- Translations l, m, n along x, y, and z axis
- Linear transformations – local scaling, shear, rotation / reflection
- Perspective transformations
- Overall scaling

- Overall scaling

$$[T_s] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

$[T_s]\ [X] = [x' \quad y' \quad z' \quad s]^T$

$= [x'/s \quad y'/s \quad z'/s \quad 1]^T$

THANK YOU