Computer Aided Geometric Design

Curves

Bezier Curve

Instructor: Dr. Prashant K. Jain
Associate Professor ME Discipline
PDPM Indian Institute of Information Technology,
Design and Manufacturing Jabalpur, Jabalpur, INDIA
Resources: web.iiitdmj.ac.in/~pkjain/
Email: pkjain@iiitdmj.ac.in, pkjain2006@gmail.com
http://www.iiitdmj.ac.in/Faculty/pkjain.html
http://in.linkedin.com/in/pkjain2006
https://www.facebook.com/pkjain2006

PDPM
INDIAN INSTITUTE OF INFORMATION TECHNOLOGY,
DESIGN AND MANUFACTURING JABALPUR
(An Institute of National Importance (INI) established by MHRD, Govt. of INDIA)

Parametric Representation of Free-form Curves

Interpolation Curves

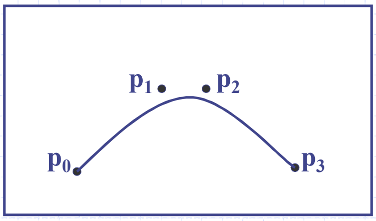
- Parametric Cubic Curve

Approximation Curves

- Bezier Curve
- B-Spline Curve
- NURBS Curve

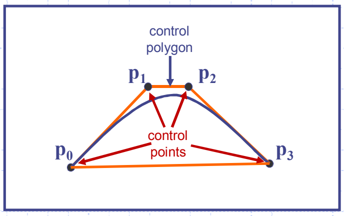
Bezier Curve

- Bezier curve is an approximation curve
- The curve was first proposed in 60's by P. Bezier
- The curve was first used to define sculptured surfaces of automobile bodies
- A cubic Bezier curve is defined by four control points



Bezier Curve

- Basis functions are real
- Degree of polynomial is one less than the number of defining polygon points
- Curve generally follows the shape of the defining polygon



Cubic Bezier Curve

Input to Cubic Bezier Curve

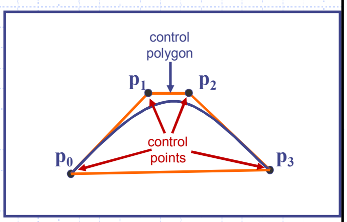
first point, $p_0 = (x_0, y_0, z_0)$

second point, $p_1 = (x_1, y_1, z_1)$

third point, $p_2 = (x_2, y_2, z_2)$

fourth point, $p_3 = (x_3, y_3, z_3)$

- *First point is starting point*
- *Fourth point is end point*
- *Order of points is important.*



Bezier Curve

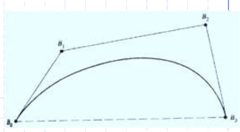
A parametric Bezier curve is mathematically defined as:

$$P(u) = \sum_{i=0}^{n} B_i J_{n,i}(u) \qquad 0 \le u \le 1$$

Where

$$J_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

$$\binom{n}{i} = {}^nC_i = \frac{n!}{i!\,(n-i)!}$$

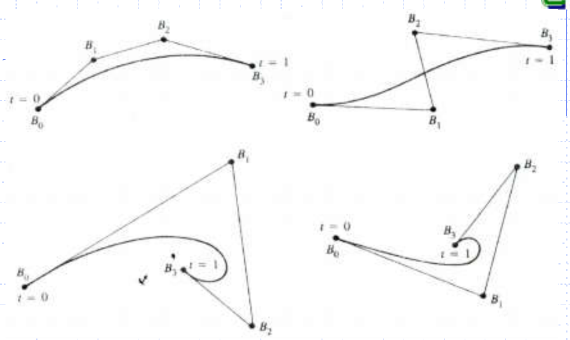


A Bezier curve and its defining polygon

$J_{n,i}(u)$ is the i[th], n[th] order Bernstein basis function.

Polynomial curve segment, is less one than the number of points in the defining Bezier polygon

Vertices are numbered from 0 to n also $(0)^0=1$ and $0!=1$



Bezier polygons for cubics

Cubic Bezier Curve

Definition

$p(u) = {}^3c_0\,(1-u)^3\,p_0 + {}^3c_1\,(1-u)^2\,u\,p_1 + {}^3c_2\,(1-u)\,u^2\,p_2 + {}^3c_3\,u^3\,p_3$

$p(u) = (1-u)^3\,p_0 + 3\,(1-u)^2\,u\,p_1 + 3\,(1-u)\,u^2\,p_2 + u^3\,p_3$

$0 \le u \le 1$

Definition

$x\,(u) = (1-u)^3\,x_0 + 3\,(1-u)^2\,u\,x_1 + 3\,(1-u)\,u^2\,x_2 + u^3\,x_3$

$y\,(u) = (1-u)^3\,y_0 + 3\,(1-u)^2\,u\,y_1 + 3\,(1-u)\,u^2\,y_2 + u^3\,y_3$

$z\,(u) = (1-u)^3\,z_0 + 3\,(1-u)^2\,u\,z_1 + 3\,(1-u)\,u^2\,z_2 + u^3\,z_3$

$0 \le u \le 1$

Bezier Curve

Cubic Bezier Curve

$$x\,(u) = \sum_{i=0}^{i=3} {}^3c_i\,(1-u)^{3-i}\,u^i\,x_i$$

Quintic Bezier Curve

$$x\,(u) = \sum_{i=0}^{i=5} {}^5c_i\,(1-u)^{5-i}\,u^i\,x_i$$

Quartic Bezier Curve

$$x\,(u) = \sum_{i=0}^{i=4} {}^4c_i\,(1-u)^{4-i}\,u^i\,x_i$$

Generic Bezier Curve

$$x\,(u) = \sum_{i=0}^{i=n} {}^nc_i\,(1-u)^{n-i}\,u^i\,x_i$$

Bezier Curve

The equation of Bezier curve can also be expressed in a matrix form

$x\,(u) = (1-u)^3\,x_0 + 3\,(1-u)^2\,u\,x_1 + 3\,(1-u)\,u^2\,x_2 + u^3\,x_3$

$x\,(u) = (1 - 3u + 3u^2 - u^3)\,x_0 + (3u - 6u^2 + 3u^3)\,x_1 + (3u^2 - 3u^3)\,x_2 + u^3\,x_3$

$x\,(u) = (-x_0 + 3x_1 - 3x_2 + x_3)\,u^3 + (3x_0 - 6x_1 + 3x_2)\,u^2 + (-3x_0 + 3x_1)\,u + x_0$

$$x\,(u) = [u^3\ u^2\ u\ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The equation of Bezier curve can also be expressed in a matrix form

$$p\,(t) = TNG = FG$$

Here $F = [J_{n_{1_0}}\ \ J_{n_{1_1}}\ \ J_{n_{1_2}} \cdots\cdots J_{n_{1_n}}]$

and $[G]^T = [\ \ B_0\ \ B_1 \cdots\cdots B_n]$

For four defining polygon points (n=3) cubic Bezier curve is

$$p\,(t) = [(1-t)^3 \quad 3t(1-t)^2 \quad 3t^2(1-t) \quad t^3] * \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Further $p\,(t) = TNG$

$$p\,(t) = [t^3 \quad t^2 \quad t \quad 1\,] * \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Similarly the quartic (n=4)

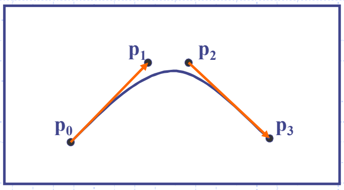
$$p\,(t) = [t^4 \quad t^3 \quad t^2 \quad t \quad 1\,] * \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

Properties of Bezier Curve

- The basis functions are real
- The degree of the polynomial defining the curve segment is one less than the number of defining polygon points
- The curve generally follows the shape of the defining polygon
- The first and last points on the curve are coincident with the first and last points of the defining polygon
- The tangent vectors at the ends of the curve have the same direction as the first and last polygon spans, respectively
- The curve is contained with in the convex hull of the defining polygon i.e. within the largest convex polygon defined by the polygon vertices. In Fig, the convex hull is shown by the polygon and the dashed line.
- The curve exhibits the variation diminishing property. Basically this means that the curve does not oscillate about any straight line more often than the defining polygon
- The curve is invariant under an affine transformation

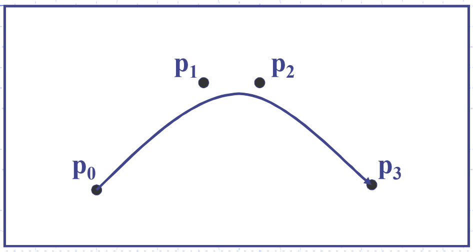
Properties of Bezier Curve

- First and last point on the curve are coincident with first and last point of defining polygon vertices.
- Tangent vectors at the ends of the curve have the same direction as the first and last polygon span.



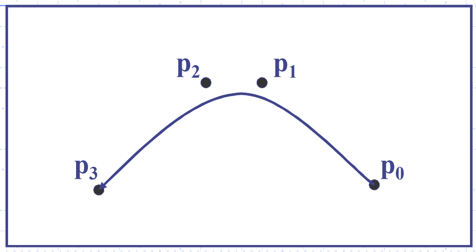
Tangents at end points are defined by end points and their adjacent points

Properties of Bezier Curve



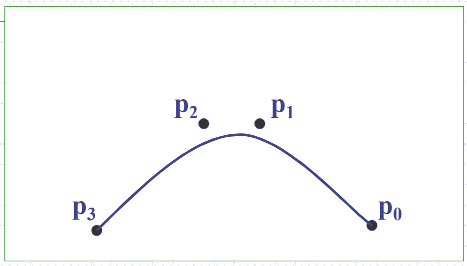
Reversing the sequence of control points does not change the shape of curve

Properties of Bezier Curve



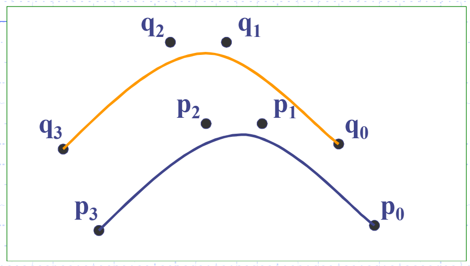
Reversing the sequence of control points does not change the shape of curve

Properties of Bezier Curve



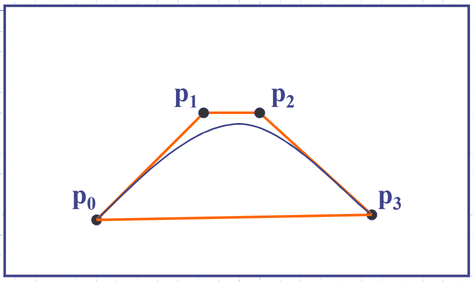
The curve is invariant under an affine transformation

Properties of Bezier Curve



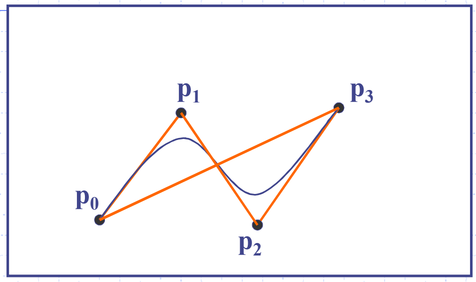
The curve is invariant under an affine transformation

Properties of Bezier Curve



Convex Hull Property

Properties of Bezier Curve



Convex Hull Property

Properties of Bezier Curve

Partition of Unity Property

$$\sum_{i=0}^{i=n} {}^nc_i\,(1-u)^{n-i}\,u^i = 1$$

For i = 3,

$(1-u)^3 + 3\,(1-u)^2\,u + 3\,(1-u)\,u^2 + u^3 = 1$

when u = 0.6

0.064 + 3 * 0.16 * 0.6 + 3 * 0.4 * 0.36 + 0.216

0.064 + 0.288 + 0.432 + 0.216

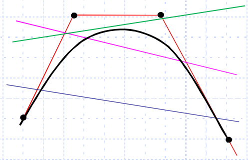
1.0

Properties of Bezier Curve

Variation diminishing property

The curve exhibits the variation diminishing property. Basically this means that the curve does not oscillate about any straight line more often than the defining polygon

No straight line intersects a Bezier curve more times than it intersects its control polygon.

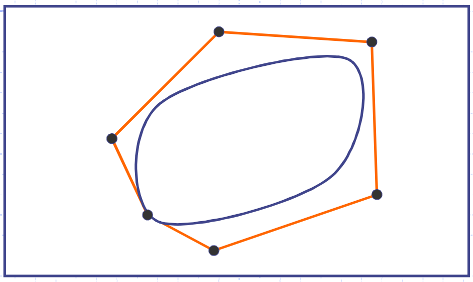


Intersections	
Curve: 0	Polygon: 2
Curve: 1	Polygon: 2
Curve: 2	Polygon: 2

For a three dimensional Bezier curve, replaces the words straight line with the word plane

Properties of Bezier Curve

Closed Curves can be defined



Closed Bezier Curve

Question

Given B0 [1 1], B1 [2 3], B2 [4 3] and B3 [3 1] the vertices of a Bezier polygon determine seven points on the Bezier curve

$$P(t) = \sum_{i=0}^{n} B_i\, J_{n,i}(t)$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Here n=3, since there are four vertices. Hence

$$\binom{n}{i} = \binom{3}{i} = \frac{6}{i!(3-i)!}$$

$$J_{3,0}(t) = t^0(1-t)^3$$

$$J_{3,1}(t) = 3t(1-t)^2$$

$$J_{3,2}(t) = 3t^2(1-t)$$

$$J_{3,3}(t) = t^3$$

Thus, $P(t) = B_0 J_{3,0} + B_1 J_{3,1} + B_2 J_{3,2} + B_3 J_{3,3}$

A table of $J_{n,i}$ for various values of t is given below

The points on the curve are then

$P(0) = B_0 = [1 \quad 1]$

$P(0.15) = 0.614B_0 + 0.325B_1 + 0.058B_2 + 0.003B_3 = [1.5 \quad 1.765]$

$P(0.35) = 0.275B_0 + 0.444B_1 + 0.239B_2 + 0.042B_3 = [2.248 \quad 2.367]$

$P(0.5) = 0.125B_0 + 0.375B_1 + 0.375B_2 + 0.125B_3 = [2.5 \quad 2.5]$

$P(0.65) = 0.042B_0 + 0.239B_1 + 0.444B_2 + 0.275B_3 = [3.122 \quad 2.367]$

$P(0.85) = 0.003B_0 + 0.058B_1 + 0.325B_2 + 0.614B_3 = [3.248 \quad 1.765]$

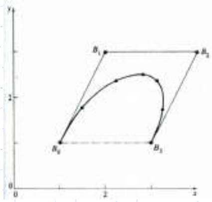
$P(1) = B_3 = [3 \quad 1]$

Coefficients of Bezier curve

$J_{n,i}$ for various values of t is given below.

t	$J_{3,0}$	$J_{3,1}$	$J_{3,2}$	$J_{3,3}$
0	1	0	0	0
0.15	0.614	0.325	0.058	0.003
0.35	0.275	0.444	0.239	0.042
0.5	0.125	0.375	0.375	0.125
0.65	0.042	0.239	0.444	0.275
0.85	0.003	0.058	0.325	0.614
1	0	0	0	1

These points are shown along with the defining polygon



Bezier Curve

Mathematically a parametric Bezier curve is defined by

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t)$$

where the Bezier or Bernstein basis or blending function is

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

With

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

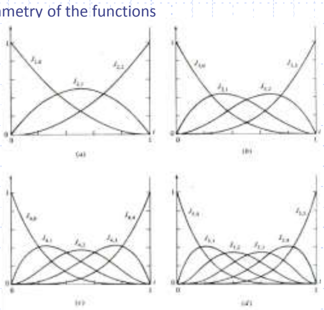
$J_{n,i}(t)$ is the i-th nth order Bernstein basis function, here n is the degree of the defining Bernstein basis function.

Thus degree of the polynomial curve segment, is one less than the number of points in the defining Bezier polygon.

Blending functions

Blending functions for several values of n

Notice the symmetry of the functions



(a) three polygon points n=2; (b) four polygon points, n=3; (c) five polygon points, n=4; (d) six polygon points, n=5.

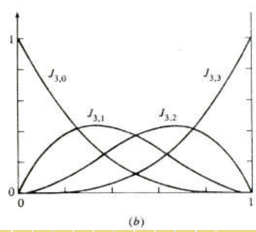
For example, each of the four blending functions shown in Figure for n=3 is a cubic. The maximum value of each blending function occurs at *t =i/n*.

$$J_{n,i}(t) = \binom{n}{i} \frac{i^i (n-i)^{n-i}}{n^n}$$

For example, for a cubic n=3, The maximum values for $J_{3,1}$ and $J_{3,2}$ occur at *1/3* and *2/3*, respectively, with values

$$J_{3,1}\left(\frac{1}{3}\right) = \frac{4}{9}$$

$$J_{3,2}\left(\frac{2}{3}\right) = \frac{4}{9}$$



Examining eqns. for the first point on the curve i.e. at t=0 shows that

$$J_{n,0}(0) = \frac{n!\,(0)^i(1-0)^{n-0}}{n!\,(n-i)!} = 1 \qquad i = 0$$

$$J_{n,i}(0) = \frac{n!\,(0)^i(1-0)^{n-i}}{n!\,(n-i)!} = 1 \qquad i \neq 0$$

Thus $\mathrm{P}(0) = B_0 J_{n,0}(0) = B_0$

This shows that the first point on the Bezier curve and on its defining polygon are coincident .

Similarly for the last point on the curve, i.e. at t=1

$$J_{n,n}(1) = \frac{n!\,(1)^i(0)^{n-n}}{n!\,(1)} = 1 \qquad i = n$$

$$J_{n,i}(1) = \frac{n!\,(t)^i(1-1)^{n-i}}{n!\,(n-i)!} = 0 \qquad i \neq n$$

Thus $\mathrm{P}(1) = B_n J_{n,n}(1) = B_n$

This shows that the last point on the Bezier curve and the last point on its defining polygon are coincident.

Derivatives Bezier Curves

Although it is not necessary to numerically specify the tangent vectors at the ends of an individual Bezier curve, maintaining slope and curvature continuity when joining Bezier curves, determining surface normal for lighting or numerical control tool path calculations, or local curvature for smoothness or fairness calculations requires a knowledge of both first and second derivatives of a Bezier curve.

Recalling equation, the first derivative of a Bezier curve is

$$\mathrm{P}'(\mathrm{t}) = \textstyle\sum_{i=0}^{n} B_i J'_{n,i}(t)$$

Second derivative is given by

$$\mathrm{P}''(\mathrm{t}) = \sum_{i=0}^{n} B_i J''_{n,i}(t)$$

Derivatives Bezier Curves

The derivatives of the basis function are obtained by formally differentiating equation $J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$. Specifically

$$J'_{n,i}(t) = \binom{n}{i}\{i\,t^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1}\}$$

$$= \binom{n}{i} t^i(1-t)^{n-i}\left\{\frac{i}{t} - \frac{(n-i)}{(1-t)}\right\}$$

$$= \frac{(i-nt)}{t(1-t)} J_{n,i}(t)$$

Similarly the second derivative is

$$J''_{n,i}(t) = \left\{\frac{(i-nt)^2 - nt^2 - i(i-2t)}{t^2\,(1-t)^2}\right\} J_{n,i}(t)$$

Derivatives Bezier Curves

At the beginning and the ends of a Bezier curve i.e. at t=0 and t=1 numerical evaluation of above equations creates difficulties.

An alternate evaluation for the rth derivative at t=0 is given by

$$P^r(0) = \frac{n!}{(n-r)!} \textstyle\sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} B_i$$

At t=1 by

$$P^r(1) = \frac{n!}{(n-r)!} \textstyle\sum_{i=0}^{r} (-1)^{i} \binom{r}{i} B_{n-i}$$

Thus the first derivatives at the ends are

$$\mathrm{P}'(0) = n(B_1 - B_0)$$

And

$$\mathrm{P}'(1) = n(B_n - B_{n-1})$$

This illustrates that the tangent vector for a Bezier curve at the initial and final points has the same direction as the initial and final polygon spans.

Derivatives Bezier Curves

Similarly the second derivatives at the ends are

$$\mathrm{P}''(0) = n(n-1)(B_0 - 2B_1 + B_2)$$

And

$$\mathrm{P}''(1) = n(n-1)\,(B_n - 2B_{n-1} + B_{n-2})$$

Thus the second derivative of the Bezier curve at the initial and final points depends on the two nearest polygon spans i.e. on the nearest three polygon vertices.

Example on Derivatives

Consider the four points Bezier polygon

$$P(t) = B_0 J_{3,0}(t) + B_1 J_{3,1}(t) + B_2 J_{3,2}(t) + B_3 J_{3,3}(t)$$

Hence first derivative is

$$P'(t) = B_0 J'_{3,0}(t) + B_1 J'_{3,1}(t) + B_2 J'_{3,2}(t) + B_3 J'_{3,3}(t)$$

Second derivative is

$$P''(t) = B_0 J''_{3,0}(t) + B_1 J''_{3,1}(t) + B_2 J''_{3,2}(t) + B_3 J''_{3,3}(t)$$

Differentiating the basis functions directly yields

$$J_{3,0}(t) = t^0(1-t)^3 \rightarrow J'_{3,0}(t) = -3(1-t)^2 \rightarrow J''_{3,0}(t) = 6(1-t)$$

$$J_{3,1}(t) = 3t(1-t)^2 \rightarrow J'_{3,1}(t) = 3(1-t)^2 - 6t(1-t) \rightarrow J''_{3,1}(t) = -6(2-3t)$$

$$J_{3,2}(t) = 3t^2(1-t) \rightarrow J'_{3,2}(t) = 6t(1-t) - 3(t)^2 \rightarrow J''_{3,2}(t) = 6(1-3t)$$

$$J_{3,3}(t) = t^3 \rightarrow J'_{3,3}(t) = 3(t)^2 \rightarrow J''_{3,3}(t) = 6t$$

Example on Derivatives

Evaluating the results at t=0 yields

$$J'_{3,0}(0) = -3,\quad J'_{3,1}(0) = 3,\quad J'_{3,2}(0) = 0,\quad J'_{3,3}(0) = 0$$

Substituting yields

$$P'(0) = -3B_0 + 3B_1 = 3(-B_0 + B_1)$$

Thus the direction of the tangent vector at the beginning of the curve is the same as that of the first polygon span

At the end of the curve, t=1 and

$$J'_{3,0}(1) = 0,\quad J'_{3,1}(1) = 0,\quad J'_{3,2}(1) = -3,\quad J'_{3,3}(1) = 3$$

Substituting yields

$$P'(1) = -3B_2 + 3B_3 = 3(-B_2 + B_3)$$

Thus, the direction of the tangent vector at the end of the curve is the same as that of the last polygon span.

Example on Derivatives

Evaluating the results at t=0 yields

$$J''_{3,0}(0) = 6,\ J''_{3,1}(0) = -12,\ J''_{3,2}(0) = 6,\ J''_{3,3}(0) = 0$$

Substituting yields

$$P''(0) = 6B_0 - 12B_1 + 6B_2 = 6(B_0 - 2B_1 + B_2)$$

Thus second derivative or curvature of Bezier curve at start point depends on three nearest polygon points or two nearest polygon spans.

At the end of the curve, t=1 and

$$J''_{3,0}(1) = 0,\ J''_{3,1}(1) = 6,\ J''_{3,2}(1) = -12,\ J''_{3,3}(1) = 6$$

Substituting yields

$$P''(1) = 6B_1 - 12B_2 + 6B_3 = 6(B_1 - 2B_2 + B_3)$$

Thus second derivative or curvature of Bezier curve at end point depends on three nearest polygon points or two nearest polygon spans.

Example on Derivatives

Evaluating the results at t=0 yields

$$J'_{3,0}(0) = -3,\quad J'_{3,1}(0) = 3,\quad J'_{3,2}(0) = 0,\quad J'_{3,3}(0) = 0$$

Substituting yields

$$P'(0) = -3P_0 + 3P_1 = 3(-P_0 + P_1)$$

Thus the direction of the tangent vector at the beginning of the curve is the same as that of the first polygon span

At the end of the curve, t=1 and

$$J'_{3,0}(1) = 0,\quad J'_{3,1}(1) = 0,\quad J'_{3,2}(1) = -3,\quad J'_{3,3}(1) = 3$$

Substituting yields

$$P'(1) = -3P_2 + 3P_3 = 3(-P_2 + P_3)$$

Thus, the direction of the tangent vector at the end of the curve is the same as that of the last polygon span.

The basis functions given above along with equations can be used to evaluate the derivations along the curve. Specifically, the first derivatives are

$$J'_{3,0}(t) = \frac{(0-3t)}{t(1-t)}(1-t)^3 = -3(1-t)^2$$

$$J'_{3,1}(t) = \frac{(1-3t)}{t(1-t)}(3t)(1-t)^2 = 3(1-3t)(1-t) = 3(1-4t+3t^2)$$

$$J'_{3,2}(t) = \frac{(2-3t)}{t(1-t)}(3t^2)(1-t) = 3t(2-3t)$$

$$J'_{3,3}(t) = \frac{3(1-t)}{t(1-t)}t^3 = (3t^2)$$

Notice that there is no difficulty in evaluating these results at either t=0 or t=1.
Substituting into eqn. yields at first derivative at any point on the curve.
For example, at t=1/2,

$$P'\left(\frac{1}{2}\right) = -3\left(1-\frac{1}{2}\right)^2 B_0 + 3\left(1-\frac{3}{2}\right)\left(1-\frac{1}{2}\right)B_1 + \left(\frac{3}{2}\right)\left(2-\frac{3}{2}\right)B_2 + \frac{3}{4}B_3$$

$$= -\frac{3}{4}B_0 - \frac{3}{4}B_1 + \frac{3}{4}B_2 + \frac{3}{4}B_3 = -\frac{3}{4}(B_0 + B_1 - B_2 - B_3)$$

Similarly, the second derivatives are

$$J''_{3,0}(t) = \frac{\{(-3t)^2 - 3t^2\}}{t^2(1-t)^2}(1-t)^3 = 6(1-t)$$

$$J''_{3,1}(t) = \frac{\{(1-3t)^2 - 3t^2 - (1-2t)\}}{t^2(1-t)^2}(3t)(1-t)^2$$
$$= -6(2-3t)$$

$$J''_{3,2}(t) = \frac{\{(2-3t)^2 - 3t^2 - 2(1-2t)\}}{t^2(1-t)^2}3t^2(1-t)$$
$$= 6(1-3t)$$

$$J''_{3,3}(t) = \frac{\{(3-3t)^2 - 3t^2 - 3(1-2t)\}}{t^2(1-t)^2}t^3 = 6t$$

For t=1/2, these results yield

$$P''\left(\frac{1}{2}\right) = 6\left(1-\frac{1}{2}\right)B_0 - 6\left(2-\frac{3}{2}\right)B_1 + 6\left(1-\frac{3}{2}\right)B_2 + 3B_3$$
$$= 3B_0 - 3B_1 - 3B_2 + 3B_3 = 3(B_0 - B_1 - B_2 + B_3)$$

Continuity conditions

If one Bezier curve P(t) of degree n is defined by vertices B_i and an adjacent Bezier curve Q(s) of degree m by vertices C_i then first derivative continuity at the joint between the curves is given by

$$P'(1) = gQ'(0)$$

Where g is a scalar.

$$n(B_n - B_{n-1}) = m(C_1 - C_0)$$

$$(C_1 - C_0) = \frac{n}{m}(B_n - B_{n-1})$$

Since positional continuity is implied at the joint, $C_0 = B_n$ and

$$C_1 = \frac{n}{m}(B_n - B_{n-1}) + B_n$$

Thus the tangent vector directions at the joint are the same if the three vertices

B_{n-1}, $B_n = C_0$ and C_1 are collinear, i.e. B_n need only lie somewhere on the line between B_{n-1} and C_1.

If both direction and magnitude of the tangent vectors at the joint are to be equal then for equal degree curves (n=m), $B_n = C_0$ must be at midpoint of the line joining B_{n-1} and C_1 i.e.

$$(C_1 - C_0) = B_n - B_{n-1} = C_0 - B_{n-1}$$

$$C_1 + B_{n-1} = 2C_0 = 2B_n$$

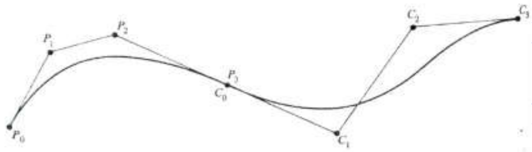


Figure illustrates this for n = m = 3 i.e. For tow cubic Bezier curves.

Second derivative continuity between adjacent Bezier curves is given by

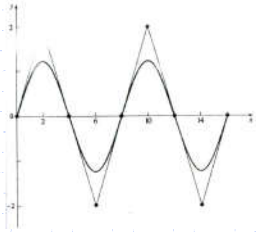
$$m(m-1)\,(C_0 - 2C_1 + C_2) = n(n-1)(B_{n-2} - 2B_{n-1} + B_n)$$

Using the conditions for C0 and C1 continuity at the joint given above yields

$$C_2 = \frac{n(n-1)}{m(m-1)} B_{n-2} - 2\left\{\frac{n(n-1)}{m(m-1)} + \frac{n}{m}\right\} B_{n-1} + \left\{1 + 2\frac{n}{m} + \frac{n(n-1)}{m(m-1)}\right\} B_n$$

For cubic Bezier curve

$$C_2 = B_{n-2} - 4(B_{n-1} - B_n)$$



Second derivative continuity between quartic Bezier curve

Additional flexibility in Bezier Curves

- Degree raising
- Sub division

Additional flexibility in Bezier curve

Degree Raising

If a Bezier curve with additional flexibility is required, the degree of the defining polynomial can be increased by increasing the number of defining polygon points.

For every point on a Bezier curve with n defining polygon vertices $B_0 \cdots B_n$ the same point on a new Bezier curve with n+1 defining polygon vertices is given by $B_0^* \cdots B^*_{n+1}$

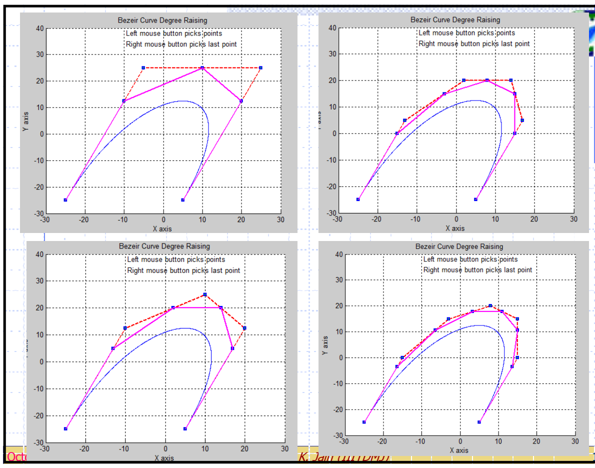
$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t) = \sum_{i=0}^{n+1} B^*_i J_{n+1,i}(t)$$

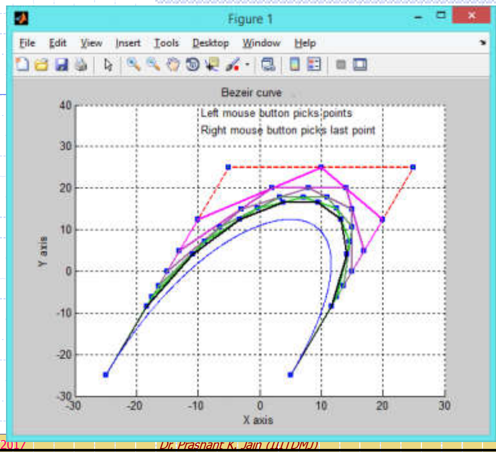
Where $B^*_0 = B_0$

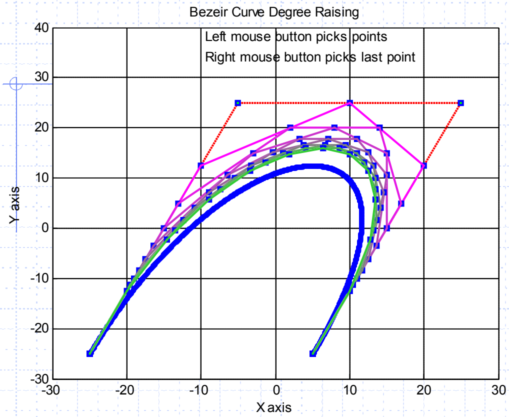
$B^*_i = \propto_i B_{i-1} + (1 - \propto_i) B_i$

$\propto_i = \frac{i}{n+1}$ $\quad i = 1, \ldots\ldots, n$

$B^*_{n+1} = B_n$







Additional flexibility in Bezier curve

Sub division

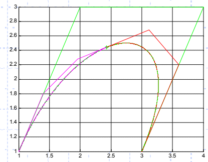
Additional flexibility can also be obtained by subdividing a Bezier curve into two new Bezier curves that, combined are identical with the original curve.

A Bezier curve can be divided at any parameter value in the range 0 to 1. The simplest choice at the midpoint i.e. t=1/2.

Here the results for midpoint subdivision are derived for the special case of cubic Bezier curves.

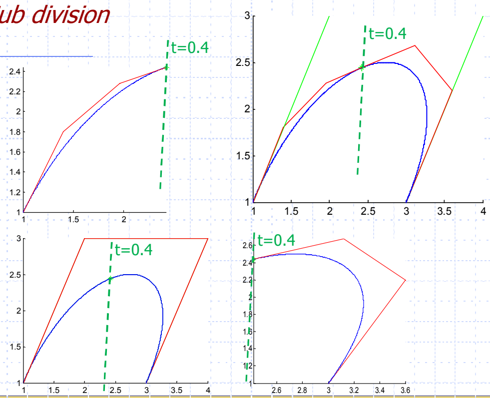
A cubic Bezier curve is given by

$$p\,(t) = (1-t)^3\, B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t)B_2 + t^3\, B_3 \qquad 0 \le t \le 1$$



Additional flexibility in Bezier curve

Sub division



Sub division of Bezier Curve

With defining polygon given by B_0, B_1, B_2, B_3. The polygon C_0, C_1, C_2, C_3 then defining the Bezier curve Q(u), $0 \le u \le 1$ corresponding to the first half of the original curve i.e. $p\,(t) = 0 \le t \le 1/2$ is required.

Similarly, the polygon D_0, D_1, D_2, D_3 that defines the Bezier curve R(v), $0 \le v \le 1$ corresponding to the second half of the original curve; $p\,(t) = 1/2 \le t \le 1$ is also required.

The new defining polygon vertices C_i *and* D_i are obtained by equating the position and tangent vectors

at $u = 0$, $t = 0$; $u = 1$, $t = 1/2$

and $v = 0$, $t = 1/2$; $v = 1$, $t = 1$;

Using Bezier definitions

$$\text{P}(t) = \textstyle\sum_{i=0}^{n} B_i\, J_{n,\,i}\,(t)$$

$$\text{P}'(t) = \textstyle\sum_{i=0}^{n} B_i\, J'_{n,\,i}\,(t)$$

We obtain these equations

$$C_0 = B_0$$

$$3(C_1 - C_0) = \tfrac{3}{2}(B_1 - B_0)$$

$$3(C_3 - C_2) = \tfrac{3}{8}(B_3 + B_2 + B_1 - B_0)$$

$$C_3 = \tfrac{1}{8}(B_3 + 3B_2 + 3B_1 + B_0)$$

Sub division of Bezier Curve

Solution of these equations gives

$$C_0 = B_0$$

$$C_1 = \tfrac{1}{2}(B_1 + B_0)$$

$$C_2 = \tfrac{1}{4}(B_2 + 2B_1 + B_0)$$

$$C_3 = \tfrac{1}{8}(B_3 + 3B_2 + 3B_1 + B_0)$$

Similarly

$$D_0 = \tfrac{1}{8}(B_3 + 3B_2 + 3B_1 + B_0)$$

$$D_1 = \tfrac{1}{4}(B_3 + 2B_2 + B_1)$$

$$D_2 = \tfrac{1}{2}(B_3 + B_2)$$

$$D_3 = B_3$$

Results generalize to

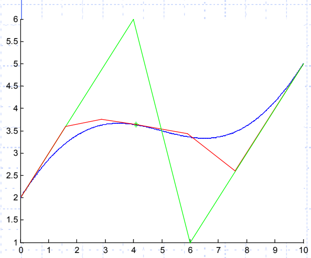
$$C_i = \textstyle\sum_{j=0}^{i} \binom{i}{j}\, \frac{B_j}{2^i} \qquad i = 0, 1, \ldots\ldots, n$$

$$D_i = \textstyle\sum_{j=i}^{n} \binom{n-i}{n-j}\, \frac{B_j}{2^{n-i}} \qquad i = 0, 1, \ldots\ldots, n$$

Applied successfully, the defining polygons converge to the Bezier curve itself

Sub division of Bezier Curve

Define a Bezier curve with four polygon vertices B_0[0 2 4], B_1[4 6 4], B_2[6 1 4] and B_3[10 5 4], then Subdivide this curve in two separate Bezier curves at t=0.4. Plot two new control polygons.



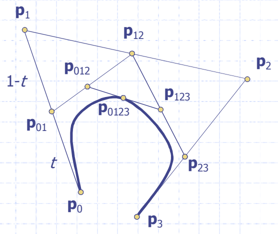
First Control Polygon		
0	2	4
1.6	3.6	4
2.88	3.76	4
4.096	3.632	4

Second Control Polygon		
4.096	3.632	4
5.92	3.44	4
7.6	2.6	4
10	5	4

de Casteljau's Algorithm

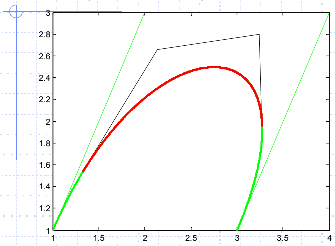
Curve Subdivision

- Cascading lerps
  - $p_{01} = (1\text{-}t)\ p_0 + t\ p_1$
  - $p_{12} = (1\text{-}t)\ p_1 + t\ p_2$
  - $p_{23} = (1\text{-}t)\ p_2 + t\ p_3$
  - $p_{012} = (1\text{-}t)\ p_{01} + t\ p_{12}$
  - $p_{123} = (1\text{-}t)\ p_{12} + t\ p_{23}$
  - $p_{0123} = (1\text{-}t)\ p_{012} + t\ p_{123}$
- Subdivides curve at p_{0123}
  - $p_0\ p_{01}\ p_{012}\ p_{0123}$
  - $p_{0123}\ p_{123}\ p_{23}\ p_3$
- Repeated subdivision converges to curve



Trimming of Bezier Curve

Now complete relationship between two sets of control polygons

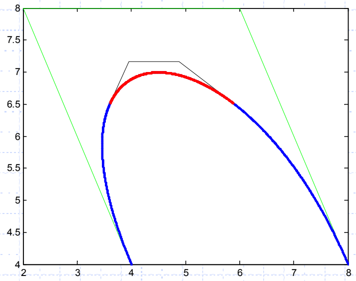


$$q_i = p_i$$
$$q_j = p_j$$
$$q_i' = \frac{u_j - u_i}{v_j - v_i} p_i'$$
$$q_j' = \frac{u_j - u_i}{v_j - v_i} p_j'$$

- tangent vector magnitudes must change to accommodate a change in the range of the parametric variable and are simply scaled by the ratio of the ranges of parametric variable.
- This preserves the direction of tangent vectors and shape of the curve.

Trimming of Bezier Curve

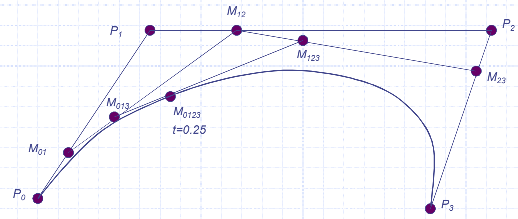
Define a Bezier curve with four polygon vertices $B_0[4\ 4]$, $B_1[2\ 8]$, $B_2[6\ 8]$ and $B_3[8\ 4]$, then Trim this curve in a smaller new Bezier curves at t=0.3 to t=0.7. Plot new control polygon and curve.



de Casteljau's Algorithm

Point on the curve

- A point on the Bezier curve for any parameter value *t* can also be find
- To find a point at *t=0.25*, instead of taking midpoints take points 0.25 of the way
- Point M_{0123} on the line M_{012}-M_{123} will be a point on the curve at t=0.25



THANK YOU