Computer Aided Geometric Design

Curves

Rational Curves and NURBS

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Rational Curves

- **Non Rational Curves:** defined by one polynomial
- **Rational Curves:** defined by the algebraic ratio of two polynomials.
- Draw their theories from perspective geometry.
- Each point is the ratio of two curves
- Just like homogeneous coordinates:

$$[x(u), y(u), z(u), w(u)] \rightarrow \left[\frac{x(u)}{w(u)}, \frac{y(u)}{w(u)}, \frac{z(u)}{w(u)}\right]$$

Advantages:

- Perspective invariant (the perspective image of rational curve is a rational curve, and can be evaluated in screen space.
- Can perfectly represent conic sections: circles, ellipses, etc.
- Piecewise cubic curve can not represent this.

Rational B-Spline curve

A Rational B-spline curve is the projection of a non rational (polynomial) B-spline curve defined in four dimensional (4D) homogenous coordinate space, back into three dimensional (3D) physical space.

Specifically:

$$P(t) = \sum_{i=1}^{n+1} B_i^h N_{i,k}(t)$$

Non rational B-spline basis function

4-D homogenous control polygon vertices for the non-rational 4-D B-spline curve

Rational B-Spline Curves

Projecting back into 3D space by dividing through by homogenous coordinate yield the rational B-spline curve.

$$P(t) = \frac{\sum_{i=1}^{n+1} B_i h_i N_{i,k}(t)}{\sum_{i=1}^{n+1} h_i N_{i,k}(t)} = \sum_{i=1}^{n+1} B_i R_{i,k}(t)$$

rational B-spline basis function

3-D control polygon vertices for the rational B-spline curve

Rational B-Spline Curves

$$R_{i,k}(u) = \frac{h_i N_{i,k}(u)}{\sum_{i=0}^{n} h_i N_{i,k}(u)}$$

$h_i > 0$ for all values of i

- Equation shows that $R_{i,k}(u)$ are a generalization of the non rational basis function $N_{i,k}(u)$.
- Substitute h_i=1: $R_{i,k}(u)=N_{i,k}(u)$.
- The rational basis function $R_{i,k}(u)$ have nearly all the analytical and geometric characteristics of their non-rational B-spline counterparts

Rational B-Spline Curves

- Each rational basis function is positive or zero for all parameter values, i.e., $R_{i,k} \geq 0$
- The sum of the rational B-spline basis for any parameter value t is one, i.e.,

$$\sum_{i=1}^{n+1} R_{i,k}(t) = 1$$

- Except for k=1, each rational basis has precisely one maximum.
- A rational B-spline curve of order k (degree k-1) is c^{k-2} continuous everywhere.
- The maximum order of rational B-spline curve is equal to the number of defining polygon vertices.
- A rational B-spline curve exhibits the variation diminishing property.
- A rational B-spline curve generally follows the shape of the defining polygon.
- A rational B-spline curve lies within the union of convex hulls formed by k successive defining polygon vertices.
- Any projective transformation is applied to a rational B-spline curve by applying it to the defining polygon vertices; i.e., the curve is invariant with respect to a projective transformation. that this is a stronger condition than that for a non-rational B-spline which is only invariant with respect to an affine transformation.

Difference between rational and non rational B Spline curves

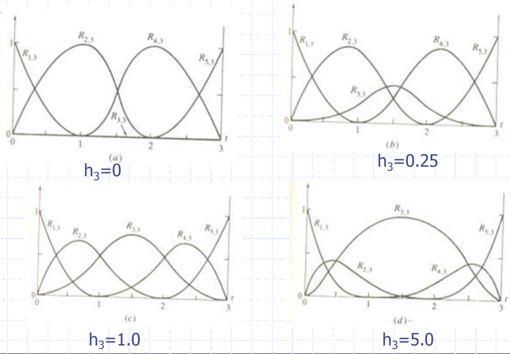
- Ability to use h_i at each control point to control the behavior of the rational curves in general.
- Choice of H vector controls the behavior of the curve.

Further Rational B-spline are:

- Unified representation that can define a variety of curves and surfaces including conics.
- Can represent all wireframe, surface and solid entities, this allows unification and conversion form one modeling technique to another.

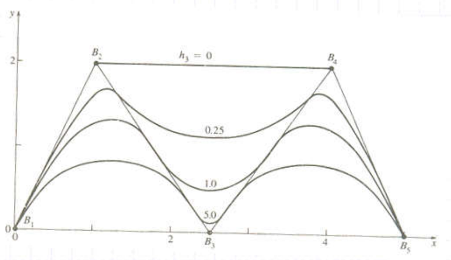
Effect of the homogeneous coordinates h on the rational B Spline basis functions

- For X=[0 0 0 1 2 3 3 3], n+1=5, k=3 and H=[1 1 h_3 1 1]



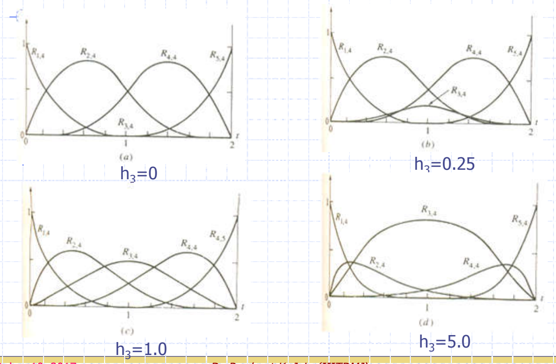
Rational B Spline Curve for

- For X=[0 0 0 1 2 3 3 3], n+1=5, k=3 and H=[1 1 h_3 1 1]



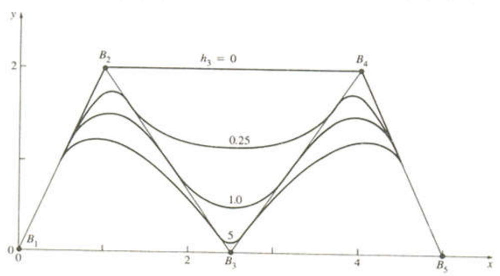
Effect of the homogeneous coordinates h on the rational B Spline basis functions

- For X=[0 0 0 0 1 2 2 2 2], n+1=5, k=4 and H=[1 1 h_3 1 1]



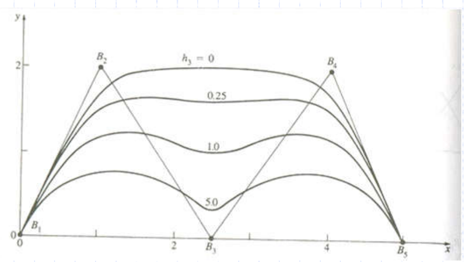
Rational B Spline curve with Periodic/uniform knot vector

- For X=[0 1 2 3 4 5 6 7], n+1=5, k=3 and H=[1 1 h_3 1 1]

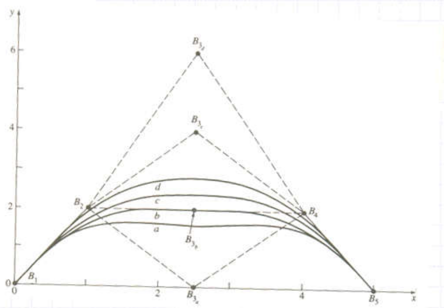


Rational B Spline Curve with open uniform knot vector

- For X=[0 0 0 0 1 2 2 2 2], n+1=5, k=3 and H=[1 1 h_3 1 1]



Effect of moving single vertex

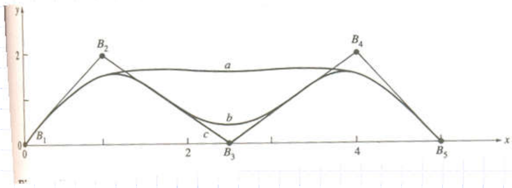


Effect of multiple vertices

a, H=[1 1 0.25 1 1]

b, H=[1 1 0.25 0.25 1 1]

c, H=[1 1 0.25 0.25 0.25 1 1]



derivatives of Rational B Spline

The derivatives of Rational B Spline curves can be obtained by formal differentiation. Specifically:

$$P'(t) = \sum_{i=1}^{n+1} B_i R'_{i,k}(t)$$

$$R'_{i,k}(t) = \frac{h_i N'_{i,k}(t)}{\sum_{i=1}^{n+1} h_i N_{i,k}} - \frac{h_i N_{i,k} \sum_{i=1}^{n+1} h_i N'_{i,k}}{\left(\sum_{i=1}^{n+1} h_i N_{i,k}\right)^2})$$

Evaluating these results at t=0 and t=n-k+2 yields

$$P'(0) = (k-1)\frac{h_2}{h_1}(B_2 - B_1)$$

$$P'(n-k+2) = (k-1)\frac{h_n}{h_{n+1}}(B_{n+1} - B_n)$$

Representation of conic sections

Provide a single mathematical description capable of blending the conic sections in to free form curves.

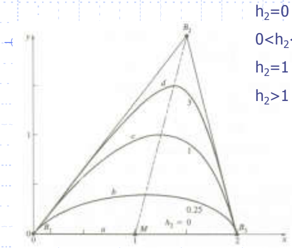
Consider a quadratic rational B spline defined by three vertices with knot vector X=[0 0 0 1 1 1] and writing this as:

$$P(t) = \frac{h_1 N_{1,3}(t) B_1 + h_2 N_{2,3}(t) B_2 + h_3 N_{3,3}(t) B_3}{h_1 N_{1,3}(t) + h_2 N_{2,3}(t) + h_3 N_{3,3}(t)}$$

Assume $h_1 = h_3 = 1$

$$P(t) = \frac{N_{1,3}(t) B_1 + h_2 N_{2,3}(t) B_2 + N_{3,3}(t) B_3}{N_{1,3}(t) + h_2 N_{2,3}(t) + N_{3,3}(t)}$$

Conic sections



At t=1/2

$h_2=0$	a straight line results.
$0<h_2<1$	an elliptic curve segment results.
$h_2=1$	a parabolic curve segment results.
$h_2>1$	a hyperbolic curve segment results.

$$P(t) = \frac{(1-t)^2 B_1 + 2h_2 t(1-t) B_2 + t^2 B_3}{(1-t)^2 + 2h_2 t(1-t) + t^2}$$

For t=1/2, P(t)=S which yields

$$S = \frac{1}{1+h_2}\frac{B_1 + B_3}{2} + \frac{h_2}{1+h_2} B_2$$

$$S = \frac{M}{1+h_2} + \frac{h_2}{1+h_2} B_2$$

writing the parametric equation of the straight line between M and B_2 gives

S=(1-u)M+uB₂

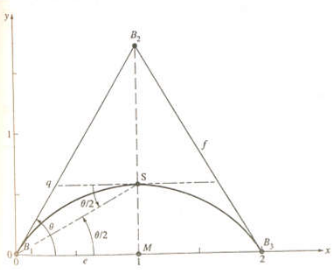
where u is the parameter.

Equating coefficients

$$u = \frac{h_2}{1+h_2} \quad and \quad h_2 = \frac{u}{1-u} = \frac{MS}{SB_2}$$

The parameter *u* controls the shape of the curve and its conic form. Hence, it is a good design tool.

Circular arc



$$h_2 = \frac{M-S}{S-B_2} = \frac{e\tan\left(\frac{\theta}{2}\right)}{f\sin\theta - e\tan\left(\frac{\theta}{2}\right)}$$

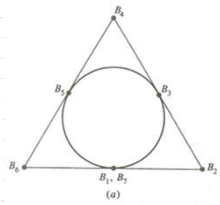
$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{(1+\cos\theta)} \quad \textit{yields}$$

$$h_2 = \frac{\frac{e\sin\theta}{1+\cos\theta}}{f\sin\theta - \frac{e\sin\theta}{1+\cos\theta}}$$

$$h_2 = \frac{e}{f(1+\cos\theta)-e} = \frac{e}{f} = \cos\theta$$

Circle

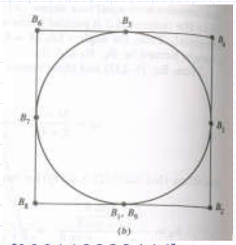
3 segments of 120^0



X=[0 0 0 1 1 2 2 3 3 3]

H=[1 1/2 1 1/2 1 1/2 1]

4 segments of 90^0



X=[0 0 0 1 1 2 2 3 3 4 4 4]

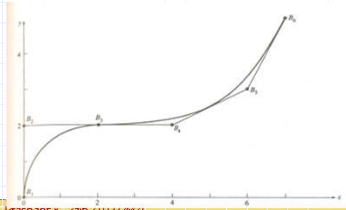
H=[1 1/√2 1 1/√2 1 1/√2 1 1/√2 1]

Generating conic sections and blending with free form curves

Construct a single third-order rational B-spline curve that blends a 90^0 circular arc defined by a qudratic rational B-spline curve with polygon vertices B_1[0 0], B_2[0 2], B_3[2 2] with the third-order quadratic rational B-spline curve defined by B_3[2 2], B_4[4 2], B_5[6 3], B_6[7 5] with hi=1, 3≤i ≤6.

The 90^0 circular arc has knot vector [0 0 0 1 1 1] and homogenous coordinate vector [1 √2/2 1]. The rational B-spline curve defined by B_3, B_4, B_5, B_6 has knot vector [0 0 0 1 2 2 2] with homogenous coordinate vector [1 1 1 1].

[X]=[0 0 0 1 1 2 3 3 3] is the nonuniform knot vector for the combined curve, with [1 √2/2 1 1 1 1] the homogenous coordinate vector.



What is NURBS

- **Non-uniform:** The amount of parameter, u, that is used to model each curve segment varies.
- **Rational:** The parametric functions that define each curve segment are ratios of polynomials:

$$x(u) = \frac{X(u)}{W(u)}, y(u) = \frac{Y(u)}{W(u)}, z(u) = \frac{Z(u)}{W(u)}$$

Advantages

- Non-uniformity permits either C^2, C^1 or C^0 continuity at join points between curve segments.
- Non-uniformity permits control points to be added to middle of curve.
- Rational curves are invariant under rotation, scaling, translation and perspective transformations of the control points (non-rational curves are not preserved under perspective projection).
  - This means we can transform the control points and redraw the curve using the transformed points.
  - If this weren't true we have to sample curve to many points and transform each point individually.

THANK YOU