Computer Aided Geometric Design

Curves

Parametric cubic curve

Instructor: Dr. Prashant K. Jain

Associate Professor ME Discipline

PDPM Indian Institute of Information Technology, Design and Manufacturing Jabalpur, Jabalpur, INDIA

Resources: web.iiitdmj.ac.in/~pkjain/

Email: pkjain@iiitdmj.ac.in, pkjain2006@gmail.com

http://www.iiitdmj.ac.in/Faculty/pkjain.html

http://in.linkedin.com/in/pkjain2006

https://www.facebook.com/pkjain2006

PDPM
INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, DESIGN AND MANUFACTURING JABALPUR
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PARAMETRIC REPRESENTATION OF FREE-FORM CURVES

Interpolation Curves

- Parametric Cubic Curve

Approximation Curves

- Bezier Curve
- B-Spline Curve
- NURBS Curve

PARAMETRIC CUBIC CURVE

- It is also known as Hermite Curve
- It is an Interpolation Curve
- It has three different forms
  - Algebraic Form (12 algebraic coefficients)
  - Geometric Form (End points & tangent vectors)
  - Four - Point Form (Four points)

Algebraic and Geometric Form

- Parametric cubic (PC) curve segment

$$x(u) = a_{3x}u^3 + a_{2x}u^2 + a_{1x}u + a_{0x} \quad (1)$$
$$y(u) = a_{3y}u^3 + a_{2y}u^2 + a_{1y}u + a_{0y} \quad (2)$$
$$z(u) = a_{3z}u^3 + a_{2z}u^2 + a_{1z}u + a_{0z} \quad (3)$$

12 unknowns

- 12 constant coefficients called algebraic coefficients
- Two same curves of same shape have different algebraic coefficients if they occupy different position in space.
- In vector notation

$$P(u) = a_3u^3 + a_2u^2 + a_1u + a_0 \quad (4)$$

P(u) is position vector

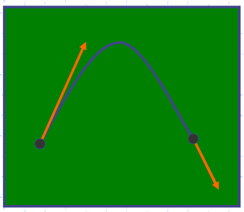
Algebraic and Geometric Form

Algebraic coefficients are not convenient for controlling shape of curve and in intuitive sense of curve

Geometric Form

starting point
(x_0, y_0, z_0)
end point
(x_1, y_1, z_1)

starting tangent vector
($x`_0$, $y`_0$, $z`_0$)
end tangent vector
($x`_1$, $y`_1$, $z`_1$)



$$x`_0 = \left.\frac{dx}{du}\right|_{u=0}$$

Algebraic and Geometric Form

- For geometric form, using two end points P(0) and P(1) and corresponding tangent vectors P'(0) and P'(1) we obtain four equations from Eq. (4).

(Substitute u = 0 & u = 1)

$$\left.\begin{aligned} p(0) &= a_0 \\ p(1) &= a_0 + a_1 + a_2 + a_3 \\ p'(0) &= a_1 \\ p'(1) &= a_1 + 2a_2 + 3a_3 \end{aligned}\right\} \quad (5)$$

- By solving

$$a_0 = p(0)$$
$$a_1 = p'(0)$$
$$a_2 = -3p(0) + 3p(1) - 2p'(0) - p'(1)$$
$$a_3 = 2p(0) - 2p(1) + p'(0) + p'(1)$$

Then eq. (4) can be written as

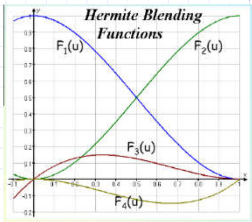
$$p(u) = (2u^3 - 3u^2 + 1)P(0) + (-2u^3 + 3u^2)P(1) + (u^3 - 2u^2 + u)P'(0) + (u^3 - u^2)P'(1)$$

Further equation can be simplified as

$$P = F_1 P_0 + F_2 P_1 + F_3 P'_0 + F_4 P'_1 \qquad \text{——————— (5)}$$

where

$$F_1 = 2u^3 - 3u^2 + 1$$
$$F_2 = -2u^3 + 3u^2$$
$$F_3 = u^3 - 2u^2 + 4$$
$$F_4 = u^3 - u^2$$



P_0, P_1, P'_0 and P'_1 are called geometric coefficients.

F terms are called blending functions also called hermite basis functions.

Algebraic equation can be written In matrix form

$$P(u) = [u^3 \quad u^2 \quad u \quad 1][a_3 \quad a_2 \quad a_1 \quad a_0]^T$$

$$P(u) = [U][A] \qquad \text{——————— (6)}$$

Similarly for geometric form

$$P(u) = [F_1 \quad F_2 \quad F_3 \quad F_4][P_0 \quad P_1 \quad P'_0 \quad P'_1]^T$$

$$= F\,B$$

Then

$$F = [(2u^3 - 3u^2 + 1) \quad (-2u^3 + 3u^2) \quad (u^3 - 2u^2 + u) \quad (u^3 - u^2)]$$

Leads to

$$F = [u^3 \quad u^2 \quad u \quad 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = U\,M_F$$

$$F = U\,M_F$$

$$P(u) = U\,M_F\,B \qquad \text{——————— (7)}$$

$$P(u) = [U][A] \qquad \text{——————— (6)}$$

$$P(u) = U\,M_F\,B \qquad \text{——————— (7)}$$

From equation 6 and 7

$$A = M_F\,B$$

and

$$B = M_F^{-1}\,A$$

where

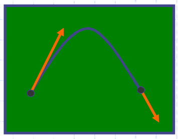
$$M_F^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

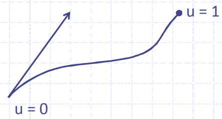
This gives conversion between algebraic and geometric forms.

Tangent Vector

- The magnitude or length of the tangent vector depends on parameterization and affects the interior shape of the curve.
- Specifying coordinates and slopes at end points of a cubic hermite curve accounts for only 10 of the 12 dof
- Six are from x_0 y_0 z_0 and x_1 y_1 z_1. Four more from direction cosines, two from each end. This means there are two more degrees of freedom to control the shape of a curve.
- We can express the matrix of geometric coefficients as

$B = [P_0 \quad P_1 \quad K_0 t_0 \quad K_1 t_1]$





PARAMETRIC CUBIC CURVE

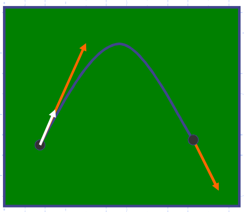
Tangent Vectors

$(x`_0, y`_0, z`_0)$
$(x`_1, y`_1, z`_1)$

can be written as

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

(l_0, m_0, n_0) & (l_1, m_1, n_1) are direction cosines of tangent vector at start & end points

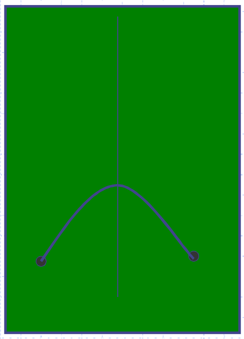


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

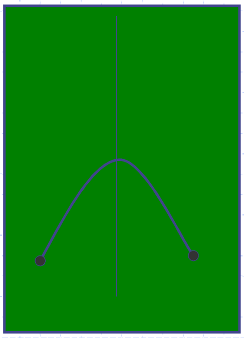


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

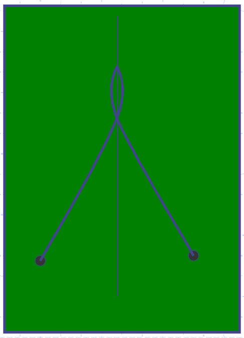


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

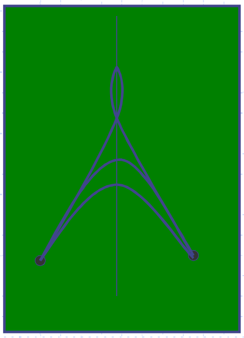


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

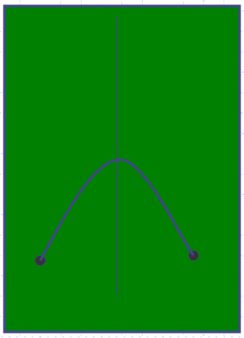


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

Effect of Increasing k_1 relative to k_0 on Curve Shape

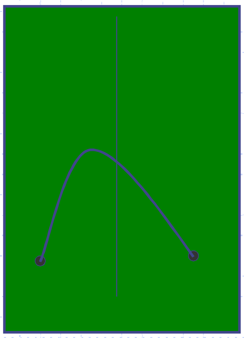


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0, k_0 m_0, k_0 n_0)$
$(k_1 l_1, k_1 m_1, k_1 n_1)$

Effect of Increasing k_1 relative to k_0 on Curve Shape

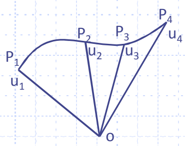


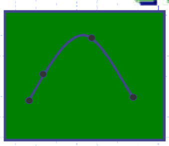
Four point form

Four point in space

$[P_1\ P_2\ P_3\ P_4]$

$u_1 < u_2 < u_3 < u_4$





Now find a matrix of constants which produces B matrix when pre multiplies with coordinate matrix of four points thus

$$[B] = K[P_1\ \ P_2\ \ P_3\ \ P_4]^T$$

$$[P_0\ \ P_1\ \ P_0'\ \ P_1'] = K[P_1\ \ P_2\ \ P_3\ \ P_4]^T$$

$$[P_1\ \ P_2\ \ P_3\ \ P_4] = [U_1\ \ U_2\ \ U_3\ \ U_4]^T MB \quad \text{From (7)}$$

where

$$U_1 = [u_1^3\ \ u_1^2\ \ u\ \ 1]$$

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$$[U_1 \quad U_2 \quad U_3 \quad U_4] = \begin{bmatrix} u_1^3 & u_1^2 & u_1 & 1 \\ u_2^3 & u_2^2 & u_2 & 1 \\ u_3^3 & u_3^2 & u_3 & 1 \\ u_4^3 & u_4^2 & u_4 & 1 \end{bmatrix}$$

For

$$B = M^{-1}[U_1 \quad U_2 \quad U_3 \quad U_4]^{T-1}[P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

Hence

$$K = M^{-1}[U_1 \quad U_2 \quad U_3 \quad U_4]^{T-1}$$

Now

$$B = K[P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

Conversely

$$[P_1 \quad P_2 \quad P_3 \quad P_4]^T = K^{-1}B$$

If we chose equally distributed u values that $u_1=0$, $u_2=1/3$, $u_3=2/3$, $u_4=1$ then K and K^{-1} will be

$$= M^{-1}U^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/27 & 1/9 & 1/3 & 1 \\ 8/27 & 4/9 & 2/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -11/2 & 9 & -9/2 & 1 \\ -1 & 9/2 & -9 & 11/2 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 20/27 & 7/27 & 4/27 & -2/27 \\ 7/27 & 20/27 & 2/27 & -4/27 \\ 0 & 1 & 0 & 0 \end{bmatrix} = U\,M$$

We can also have

let $[P_1 \quad P_2 \quad P_3 \quad P_4]^T = P$

Since p = UMB and B=KP

hence p=UMKP

Let MK = N

Hence p = UNP

$$N = \begin{bmatrix} -9/2 & 27/2 & -27/2 & 9/2 \\ 9 & -45/2 & 18 & -9/2 \\ -11/2 & 9 & -9/2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Algebraic coefficients are found from A = NP

$p = (4.5u^3 + 9u^2 - 5.5u + 1)\, P_1$

$+ (13.5u^3 - 22.5u^2 + 9u)\, P_2$

$+(-13.5u^3 + 18u^2 - 4.5u)\, P_3$

$+(4.5u^3 - 4.5u^2 + u)\, P_4$

THANK YOU