November 25, 2017 — 2:30-5:30

END-Semester Examination
ME-601: CAGD Indicative Answer Key

Duration: 180 Min. — Maximum Marks: 100

All questions are compulsory.

1. Write a procedure to truncate a parametric cubic curve segment at two specified values of u and subsequently reparametrize it. Test your formulation for a parametric cubic curve with a given set of end points $P_0(1,1,1)$ and $P_1(4,2,4)$ and the end tangents $p^u(0)=(1,1,0)$ and $p^u(1)=(1,1,1)$ truncated at : (15)

Re-Parameterization

- For a more general case curve is initially parameterized from u_i to u_j and want to change parametric variable ranges from v_i to v_j.
- Let $B_1 = [p_i \;\; p_j \;\; p_i' \;\; p_j']^T$ and $B_2 = [q_i \;\; q_j \;\; q_i' \;\; q_j']^T$
- The end points are invariant or insensitive to any change of parameterization, so $q_i = p_i$ and $q_j = p_j$ to maintain constant position.
- Tangent vectors are sensitive to the functional relationship between u and v, i.e. v = f(u), a linear relationship is required to preserve cubic form.

Thus v = au + b then dv = a du

also $v_i = au_i + b$ and $v_j = au_j + b$ from this a and b can be found

and relationship between tangent vectors

$$q' = \frac{u_j - u_i}{v_j - v_i} p'$$

November 16, 2017 — Dr. Prashant K. Jain (IIITDMJ) — 47

Re-Parameterization

Now complete relationship between two sets of geometric coefficients

$$q_i = p_i$$
$$q_j = p_j$$
$$q_i' = \frac{u_j - u_i}{v_j - v_i} p_i'$$
$$q_j' = \frac{u_j - u_i}{v_j - v_i} p_j'$$

- This also shows that tangent vector magnitudes must change to accommodate a change in the range of the parametric variable.
- Magnitudes are simply scaled by the ratio of the ranges of parametric variable.
- This preserves the direction of tangent vectors and shape of the curve.

November 16, 2017 — Dr. Prashant K. Jain (IIITDMJ) — 48

	Original Curve	Curve at u = 0.25 and u = 0.75	Curve at u = 0.333 and u = 0.667
P_0	1,1,1	1.5625 1.2500 1.4219	1.8506 1.3330 1.7025
P_1	4,2,4	3.4375 1.7500 3.3906	3.1494 1.6670 3.0754
$p^u(0)$	1,1,0	1.6250 0.5000 1.5313	1.2242 0.3340 1.2240
$p^u(1)$	1,1,1	1.6250 0.5000 1.7813	1.2242 0.3340 1.3356

2. Show that a fourth order B Spline curve with four defining polygon vertices using open uniform knot vector yields a cubic Bezier curve. (15)

Numericals: Calculating an open B-Spline Curve

For k=4 the order of the curve is equal to the number of defining polygon vertices.

Thus the B-spline curve reduces to a Bezier curve. The knot vector with t_{max} = n-k+2 = 3-4+2 =1 is [0 0 0 0 1 1 1 1].

The basis functions are:

$0 \le t < 1$

$N_{4,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 4$

$N_{3,2}(t) = (1-t)$; $N_{4,2}(t) = t$; $N_{i,2}(t) = 0$, $i \ne 3,4$

$N_{2,3}(t) = (1-t)^2$; $N_{3,3}(t) = 2t(1-t)$;

$N_{4,3}(t) = t^2$; $N_{i,3}(t) = 0$, $I \ne 2,3,4$

$N_{1,4}(t) = (1-t)^3$; $N_{2,4}(t) = t(1-t)^2 + 2t(1-t)^2 = 3t(1-t)^2$;

$N_{3,4}(t) = 2t^2(1-t) + (1-t)t^2 = 3t^2(1-t)$; $N_{4,4}(t) = t^3$;

November 25, 2017 — Dr. Prashant K. Jain (IIITDMJ) — 168

Numericals: Calculating an open B-Spline Curve

Using equation

$$p(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t) \qquad t_{min} < t \le t_{max} \qquad 2 < k \le n+1$$

The parametric B-Spline curve is

$P(t) = B_1 N_{1,4}(t) + B_2 N_{2,4}(t) + B_3 N_{3,3}(t) + B_4 N_{4,4}(t)$

$P(t) = (1-t)^3 B_1 + 3t(1-t)^2 B_2 + 3t^2(1-t) B_3 + t^3 B_4$

Thus, at t=0 $P(0) = B_1$

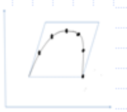
and at t= ½

$$P(\tfrac{1}{2}) = \tfrac{1}{8}B_1 + \tfrac{3}{8}B_2 + \tfrac{3}{8}B_3 + \tfrac{1}{8}B_4$$

and

$$P(\tfrac{1}{2}) = \tfrac{1}{8}[1 \;\; 1] + \tfrac{3}{8}[2 \;\; 3] + \tfrac{3}{8}[4 \;\; 3] + \tfrac{1}{8}[3 \;\; 1]$$

= [11/4 5/2]



November 25, 2017 — Dr. Prashant K. Jain (IIITDMJ) — 169

3. Write down the parametric equation of following.
   a. A straight line joining points (1, 1, 1) and (10, 2, 3)
   b. A semi circular arc of radius 6 units lying in y = 20 plane. The arc has starting point at (13, 20, 0) lies completely above x-axis with center at (7, 20, 0).
   c. Write down the parametric equation of ruled surface defined by joining above two curves.
   d. Find out coordinates of point (u = 0.5, w = 0.5) on the above surface. (15)

(a) $x(u) = 1 + 9u$
$y(u) = 1 + u$
$z(u) = 1 + 2u$
$u = 0$ to 1

(b) $x = 7 + 6\cos\theta = 7 + 6\cos(\pi u)$
$y = 20 = 20$
$z = 6\sin\theta = 6\sin(\pi u)$
$u = 0$ to 1

(c) $Q(u,w) = P(u,0)(1-w) + P(u,1)w$

$= [(1-w) \quad w] \begin{bmatrix} P(u,0) \\ P(u,1) \end{bmatrix}$

$= [1-w \quad w] \begin{bmatrix} 1+9u & 1+u & 1+2u \\ 7+6\cos(\pi u) & 20 & 6\sin(\pi u) \end{bmatrix}$

u & $w = 0$ to 1

(d) Putting $u = 0.5$ $w = 0.5$

$[6.25 \quad 10.75 \quad 4]$

4. What is an average and Gaussian curvature with respect a parametric surface. Explain, How to decide whether given surface is developbale or not. (15)

To find whether surface or a portion of a surface is developable or not curvature of parametric surface is considered.

- At any point P on a surface, the curve of intersection of a plane containing the normal to the surface at P and the surface has a curvature K.
- As the plane is rotated about the normal, the curvature changes.
- There will be unique directions for which the curvature is a minimum and a maximum exists.
- Euler the great Swiss mathematician, Showed that unique directions for which the curvature is a minimum and a maximum exists.
- The curvatures in these directions are called the principle curvatures, K_{min} and K_{max}.
- The principal curvature directions are orthogonal.
- Two combinations of the principal curvatures are of particular interest, Average and Gaussian curvatures.

$$H = \frac{\kappa_{\min} + \kappa_{\max}}{2} \qquad K = \kappa_{\min}\,\kappa_{\max}$$

- *For a developable surface the Gaussian curvature K is everywhere zero, i.e., K=0.*
- Dill has shown that for bi-parametric surfaces the average and Gaussian curvatures are given by:

$$H = \frac{A|Q_w|^2 - 2BQ_u \cdot Q_w + C|Q_u|^2}{2|Q_u \times Q_w|^3} \qquad K = \frac{AC - B^2}{|Q_u \times Q_w|^4}$$ where $$(ABC) = [Q_u \times Q_w] \cdot [Q_{uu} \quad Q_{uw} \quad Q_{ww}]$$

$K_{min}K_{max}$	K	Shape
Same Sign	> 0	Elliptic (bump or hollow)
Opposite sign	< 0	Hyperbolic (saddle point)
One or both zero	0	Cylindrical/ conical (ridge, hollow, plane)

5. Define a Bezier curve with four polygon vertices $B_1[1\ 1]$, $B_2[2\ 3]$, $B_3[4\ 3]$ and $B_4[3\ 1]$, split this curve into two curves each one being a cubic Bezier curve and find out the control points of these two curves if the original curve is split at point corresponding to parametric value u = 0.5. (20)

$P(t) = (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t) B_2 + t^3 B_3$

$P(0.5) = \frac{1}{8} B_0 + \frac{3}{8} B_1 + \frac{3}{8} B_2 + \frac{1}{8} B_3$

$= [2.75\ \ 2.5]$

$P'(t) = -3(1-t)^2 B_0 + [3(1-t)^2 - 6t(1-t)] B_1 + [6t(1-t) - 3t^2] B_2 + 3t^2 B_3$

Polygon C_0, C_1, C_2 & C_3 defines curve $Q(u)$ $0 \le u \le 1$ for first half and D_0, D_1, D_2 & D_3 defines curve $R(v)$ $0 \le v \le 1$ for second half.

equating the position and tangent vectors at $u=0, t=0$; $u=1$ $t=1/2$ and $v=0, t=1/2$; $v=1$ $t=1$

tangent vector will be half for $Q(u)$ and $R(v)$ since $P(t)$ is subdivided at $u=0.5$. This gives.

$C_0 = B_0$

$3(C_1 - C_0) = \frac{1}{2} \cdot 3(B_1 - B_0)$

$3(C_3 - C_2) = \frac{1}{2} \cdot \frac{3}{4}(B_3 + B_2 - B_1 - B_0)$

and $C_3 = \frac{1}{8}(B_3 + 3B_2 + 3B_1 + B_0)$

Solution of these equation gives.

$C_0 = B_0 = (1, 1)$

$C_1 = \frac{1}{2}(B_1 + B_0) = (1.5, 2)$

$C_2 = \frac{1}{4}(B_2 + 2B_1 + B_0) = (2.25, 2.5)$

$C_3 = \frac{1}{8}(B_3 + 3B_2 + 3B_1 + B_0) = (2.75, 2.5)$

$D_0 = \frac{1}{8}(B_3 + 3B_2 + 3B_1 + B_0) = (2.75, 2.5)$

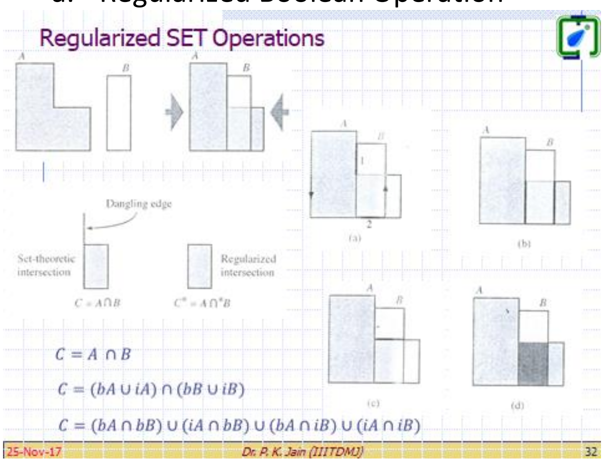
$D_1 = \frac{1}{4}(B_3 + 2B_2 + B_1) = (3.25, 2.25)$

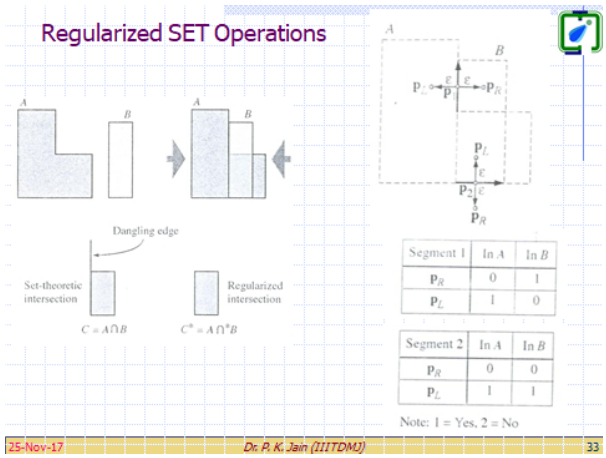
$D_2 = \frac{1}{2}(B_3 + B_2) = (3.5, 2)$

$D_3 = B_3 = (3, 1)$

6. Briefly explain following terms and phrases. (4x5)

a. Regularized Boolean Operation





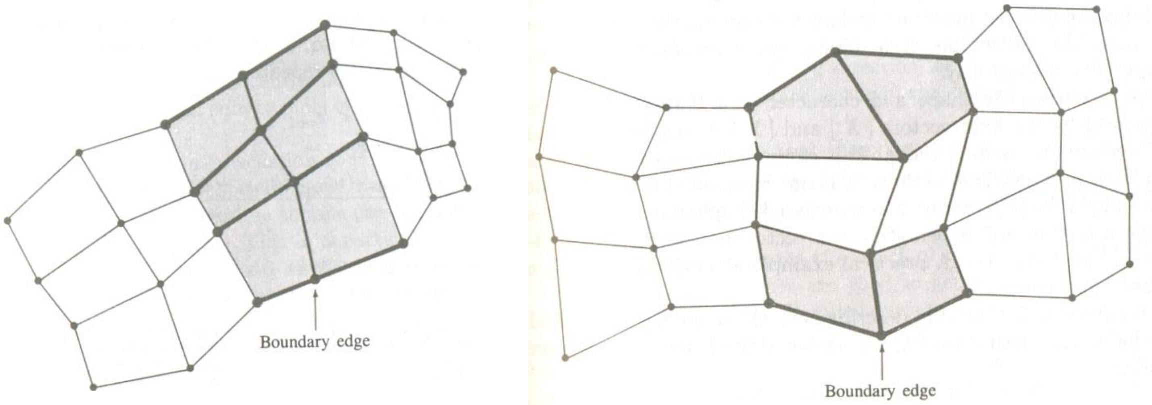
b. Explain Conditions for C^0 and C^1 continuity between two Bezier patches along a common boundary.

For C^0 continuity along the edge of two boundary curves and hence two boundary polygons along edge must be coincident.

For C^1 continuity across the patch boundary, surface normal direction along boundary edge must be same. Two conditions may be used to achieve this:

I. *Four polygon net lines that meet at and cross the boundary edge to be collinear.*

II. *Three polygon net edges meeting at the ends of the boundary curve to be coplanar.*



c. Explain use and need of the homogeneous coordinates

Homogeneous Representation

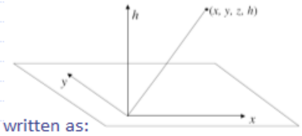
- Rotation, scaling, shearing, reflection are in form of matrix multiplication, but the translation takes the form of vector addition. This makes it inconvenient to concatenate transformation involving translation. It is desirable to express all geometric transformations in the form of matrix multiplication on representing points by their homogeneous coordinates.
- Provides an effective way to unify the description of geometric transformations as matrix multiplication.
- In homogeneous coordinates, an n-dimensional space is mapped into (n+1) dimensional space that is a point P(x,y,z) has homogeneous coordinates (x',y',z',h) where h is any scalar factor which is not equal to 0.

November 25, 2017 Dr. Prashant K. Jain (IIITDMJ) 24

Homogeneous coordinates

- A point in homogeneous coordinates (x, y, z, h), h≠0, corresponds to the 3-D vertex (x/h, y/h, z/h) in Cartesian coordinates.
- Homogeneous coordinates in 3D give rise to 4 dimensional position vector

$$x=\frac{x'}{h},\ y=\frac{y'}{h},\ z=\frac{z'}{h}$$



Translation matrices can now be written as:

$$[x' \quad y' \quad z' \quad 1]^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_d & y_d & z_d & 1 \end{bmatrix}^T * [x \quad y \quad z \quad 1]^T$$

November 25, 2017 Dr. Prashant K. Jain (IIITDMJ) 25

d. Explain the advantage of Representing curves and surfaces as NURBS

Advantage of Representing curves and surfaces as NURBS

- **Non Rational Curves:** defined by one polynomial
- **Rational Curves:** defined by the algebraic ratio of two polynomials.
- Each point is the ratio of two curves, just like homogeneous coordinates:

$$[x(u), y(u), z(u), w(u)] \rightarrow \left[\frac{x(u)}{w(u)}, \frac{y(u)}{w(u)}, \frac{z(u)}{w(u)}\right]$$

Advantages:

- Draw their theories from perspective geometry and Perspective invariant (the perspective image of rational curve is a rational curve, and can be evaluated in screen space.
- Unified representation that can define a variety of curves and surfaces including conics sections: circles, ellipses, etc. Piecewise cubic curve can not represent this.
- Can represent all wireframe, surface and solid entities, this allows unification and conversion form one modeling technique to another.
- Ability to use h, at each control point to control the behavior of the rational curves in general. Choice of H vector controls the behavior of the curve.
- Non-uniformity permits either C^2, C^1 or C^0 continuity at join points between curve segments. Non-uniformity permits control points to be added to middle of curve.

November 25, 2017 Dr. Prashant K. Jain (IIITDMJ) 174

*****