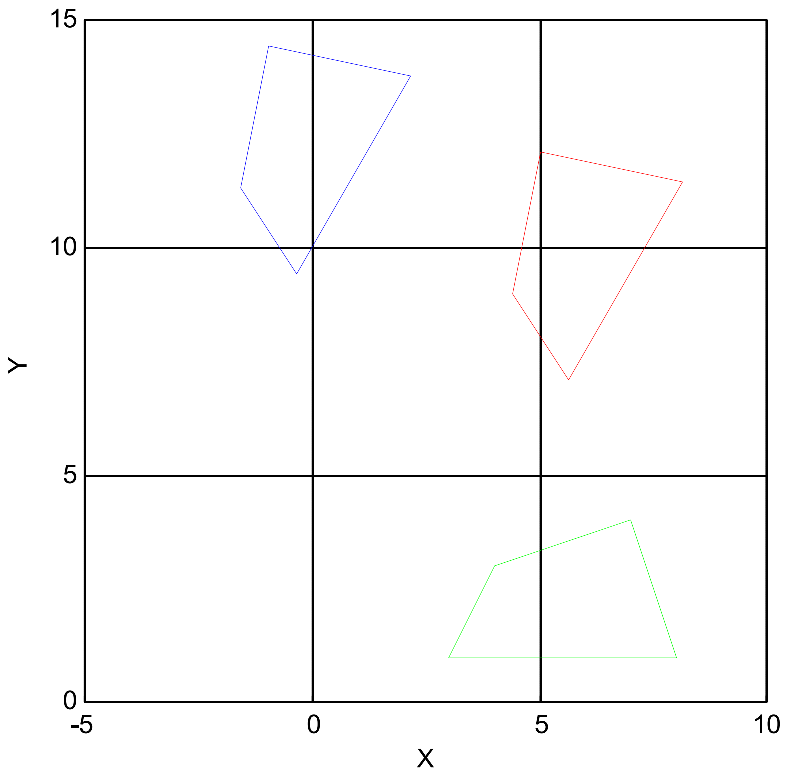
MID-Semester Examination
ME-601 CAGD : Solution

Duration: 120 Min. Maximum Marks: 60
All questions are compulsory.

1. Lamina ABCD with coordinates (4, 3), (3, 1), (8, 1) and (7, 4) respectively is first rotated through 60° about the origin and then translated by (5, 4). In another sequence, the lamina is first translated by (5, 4) and then rotated through 60° about the origin. Find the final positions and orientations of given lamina for the two sequences of transformations and show that lamina acquires different positions and orientations for the two sequences of transformations. 10

	Original		Rotation then Translation		Translation then Rotation	
Point	X	Y	X	Y	X	Y
A	4	3	4.4019	8.9641	-1.5622	11.2942
B	3	1	5.6340	7.0981	-0.3301	9.4282
C	8	1	8.1340	11.4282	2.1699	13.7583
D	7	4	5.0359	12.0622	-0.9282	14.3923



2. Explain merits and demerits of parametric representation of curves compared to implicit/explicit representation, you can give suitable examples. Explain use and need of the homogeneous coordinate system. 10

Explain with sketches

i. Parametric equations completely separate the role of dependent and independent variables
ii. Offers more degree of freedom for controlling the shape of curves and surfaces
iii. Transformations are easier to apply
iv. Advantage in representation of curve and surface segments
v. Advantage in handling infinite slope
vi. Advantage in calculation of points for display and tool path

Homogeneous Representation

- Rotation, scaling, shearing, reflection are in form of matrix multiplication, but the translation takes the form of vector addition. This makes it inconvenient to concatenate transformation involving translation. It is desirable to express all geometric transformations in the form of matrix multiplication on representing points by their homogeneous coordinates.
- Provides an effective way to unify the description of geometric transformations as matrix multiplication.
- In homogeneous coordinates, an n-dimensional space is mapped into (n+1) dimensional space that is a point P(x,y,z) has homogeneous coordinates (x',y',z',h) where h is any scalar factor which is not equal to 0.
- A point in homogeneous coordinates (x, y, z, h), $h \neq 0$, corresponds to the 3-D vertex (x/h, y/h, z/h) in Cartesian coordinates.
- Homogeneous coordinates in 3D give rise to 4 dimensional position vector

3. In case of axonometric projection derive the expression and find out the angle of rotation with respect to two principal axes to keep foreshortening ratio equal in all three directions. Also find out the foreshortening factor. 10

Isometric Projection

- Isometric Projection is a special case of trimetric projection with all three foreshortening factors equal.
- Isometric Projection is constructed by a rotation about y-axis through an angle ϕ followed by a rotation about x-axis through an angle θ and projection on z = 0 plane.
- Specific rotation angle can be obtained as:

Resulting Transformation is : $T = R_y\, R_x\, P_z$

$$[T]=\begin{bmatrix}\cos\phi & 0 & -\sin\phi & 0\\ 0 & 1 & 0 & 0\\ \sin\phi & 0 & \cos\phi & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[T]=\begin{bmatrix}\cos\phi & \sin\phi\sin\theta & 0 & 0\\ 0 & \cos\theta & 0 & 0\\ \sin\phi & -\cos\phi\sin\theta & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

September 19, 2017 Dr. Prashant K. Jain (IIITDMJ) 33

Isometric Projection

Unit vectors on the x, y and z principal axes transform to

$$[U^*]=UT=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\sin\theta & 0 & 0\\ 0 & \cos\theta & 0 & 0\\ \sin\phi & -\cos\phi\sin\theta & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[U^*]=\begin{bmatrix}\cos\phi & \sin\phi\sin\theta & 0 & 1\\ 0 & \cos\theta & 0 & 1\\ \sin\phi & -\cos\phi\sin\theta & 0 & 1\end{bmatrix}$$

$$f_x=\sqrt{x_x^{*2}+y_x^{*2}}=\sqrt{\cos^2\phi+\sin^2\phi\sin^2\theta} \quad \text{---------------------(A)}$$

$$f_y=\sqrt{x_y^{*2}+y_y^{*2}}=\sqrt{\cos^2\theta} \quad \text{---------------------(B)}$$

$$f_z=\sqrt{x_z^{*2}+y_z^{*2}}=\sqrt{\sin^2\phi+\cos^2\phi\sin^2\theta} \quad \text{---------------------(C)}$$

September 19, 2017 Dr. Prashant K. Jain (IIITDMJ) 34

Isometric Projection

Equating equation A and B

$$\cos^2\phi + \sin^2\phi\sin^2\theta = \cos^2\theta$$
$$1-\sin^2\phi + \sin^2\phi\sin^2\theta = 1-\sin^2\theta$$
$$\sin^2\phi(1-\sin^2\theta) = \sin^2\theta$$
$$\sin^2\phi = \frac{\sin^2\theta}{1-\sin^2\theta} \quad \text{-------------------------(D)}$$

We know

$\cos^2\phi = 1-\sin^2\phi$

$\cos^2\theta = 1-\sin^2\theta$

Now equate equation B and C:

$$\cos^2\theta = \sin^2\phi + \cos^2\phi\sin^2\theta$$
$$1-\sin^2\theta = \sin^2\phi + (1-\sin^2\phi)\sin^2\theta$$
$$1-\sin^2\theta = \sin^2\phi + \sin^2\theta - \sin^2\phi\sin^2\theta$$
$$1-2\sin^2\theta = \sin^2\phi(1-\sin^2\theta)$$
$$\sin^2\phi = \frac{1-2\sin^2\theta}{1-\sin^2\theta} \quad \text{-------------------------(E)}$$

We know

$\cos^2\phi = 1-\sin^2\phi$

$\cos^2\theta = 1-\sin^2\theta$

Isometric Projection

Now Equating equation D and E

$$\frac{\sin^2\theta}{1-\sin^2\theta} = \frac{1-2\sin^2\theta}{1-\sin^2\theta}$$

Gives

$$\sin^2\theta = 1/3$$
$$\sin\theta = \pm 1/\sqrt{3}$$
$$\theta = \pm 35.26^0$$

Using equation D

$$\sin^2\phi = \frac{1/3}{1-1/3} = 1/2 \qquad \phi = \pm 45^0$$

Then using equation B

$$f_z = \sqrt{\cos^2\theta} = \sqrt{2/3} = 0.8165$$

Isometric Projection

The angle: projected x-axis makes with the horizontal is important for manual construction.

Transforming the unit vector along the x-axis

$$[U_x^*] = [1 \quad 0 \quad 0 \quad 1]\begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[U_x^*] = [\cos\phi \quad \sin\phi\sin\theta \quad 0 \quad 1]$$

The angle between projected x-axis and the horizontal is then

$$\tan\alpha = \frac{y_x^*}{x_x^*} = \frac{\sin\phi\sin\theta}{\cos\phi} = \pm\sin\theta$$

Since sin ∅ = cos ∅ for ∅ = 45°, then α is

$$\alpha = \tan^{-1}(\pm\sin 35.26) = \pm 30^0$$

September 19, 2017 Dr. Prashant K. Jain (IIITDMJ) 37

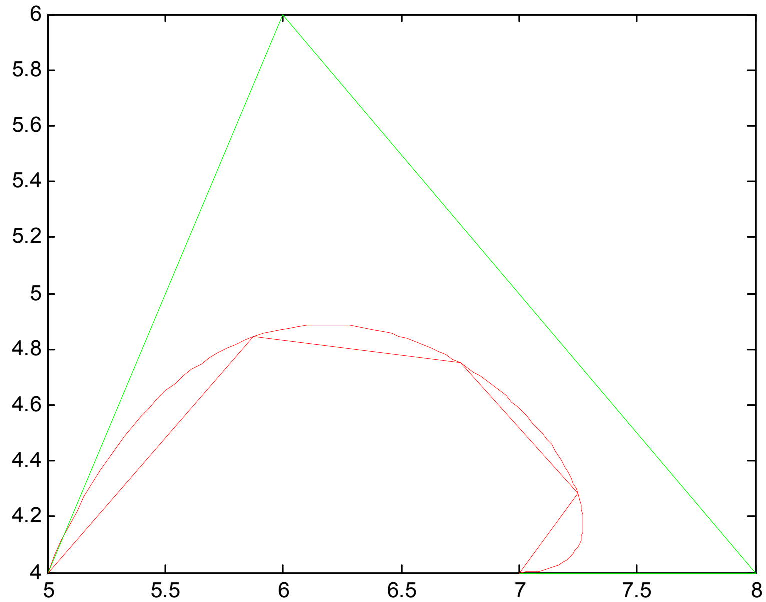
4. Discuss various properties and limitations of Bezier curves with suitable example/sketch. 10

Properties of Bezier Curve

- The basis functions are real
- The degree of the polynomial defining the curve segment is one less than the number of defining polygon points
- The curve generally follows the shape of the defining polygon
- The first and last points on the curve are coincident with the first and last points of the defining polygon
- The tangent vectors at the ends of the curve have the same direction as the first and last polygon spans, respectively
- The curve is contained with in the convex hull of the defining polygon i.e. within the largest convex polygon defined by the polygon vertices. In Fig, the convex hull is shown by the polygon and the dashed line.
- The curve exhibits the variation diminishing property. Basically this means that the curve does not oscillate about any straight line more often than the defining polygon
- The curve is invariant under an affine transformation

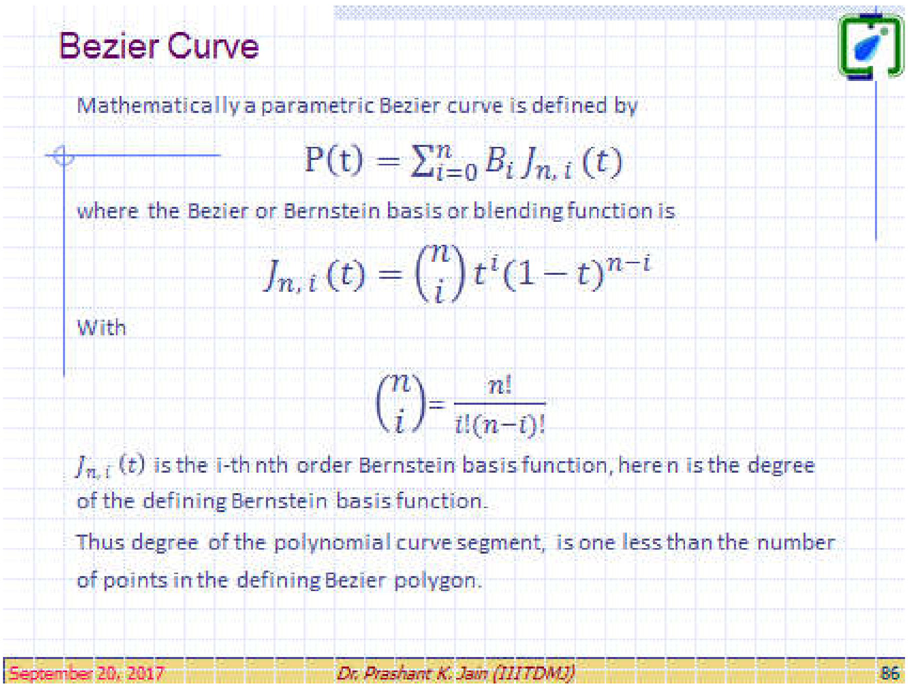
September 19, 2017 Dr. Prashant K. Jain (IIITDMJ) 70

5. Define a Bezier curve with four polygon vertices $B_0[5\ 4\ 4]$, $B_1[6\ 6\ 4]$, $B_2[8\ 4\ 4]$ and $B_3[7\ 4\ 4]$, calculate five point on this curve at equal parametric interval and plot these points on Bezier curve and control polygon. 10



5.0000	4.0000
5.8750	4.8438
6.7500	4.7500
7.2500	4.2813
7.0000	4.0000

6. Mathematically prove following with respect to Bezier curve: 10
   a. First and last point of the Bezier curve is same as the first and last point of the defining control polygon of the Bezier curve.
   b. Tangent vector for a Bezier curve at the start and end point has the same direction as the first and last polygon span.
   c. Second derivative or curvature of Bezier curve at start and end point depends on three nearest polygon points or two nearest polygon spans.



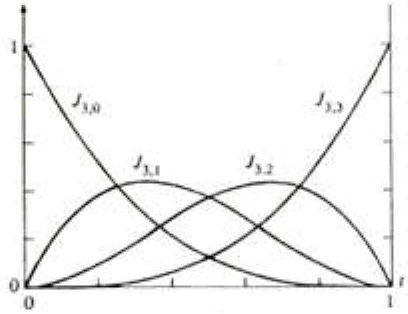
For example, each of the four blending functions shown in Figure for n=3 is a cubic. The maximum value of each blending function occurs at $t = i/n$.

$$J_{n,i}(t) = \binom{n}{i} \frac{i^i (n-i)^{n-i}}{n^n}$$

For example, for a cubic n=3, The maximum values for $J_{3,1}$ and $J_{3,2}$ occur at *1/3* and *2/3*, respectively, with values

$$J_{3,1}\left(\frac{1}{3}\right) = \frac{4}{9}$$

$$J_{3,2}\left(\frac{2}{3}\right) = \frac{4}{9}$$



(b)

Examining eqns. for the first point on the curve i.e. at t=0 shows that

$$J_{n,0}(0) = \frac{n!\,(0)^i (1-0)^{n-0}}{n!\,(n-i)!} = 1 \qquad i = 0$$

$$J_{n,i}(0) = \frac{n!\,(0)^i (1-0)^{n-i}}{n!\,(n-i)!} = 1 \qquad i \neq 0$$

Thus $\quad P(0) = B_0 J_{n,0}(0) = B_0$

This shows that the first point on the Bezier curve and on its defining polygon are coincident.

Similarly for the last point on the curve, i.e. at t=1

$$J_{n,n}(1) = \frac{n!\,(1)^i (0)^{n-n}}{n!\,(1)} = 1 \qquad i = n$$

$$J_{n,i}(1) = \frac{n!\,(t)^i (1-1)^{n-i}}{n!\,(n-i)!} = 0 \qquad i \neq n$$

Thus $\quad P(1) = B_n J_{n,n}(1) = B_n$

This shows that the last point on the Bezier curve and the last point on its defining polygon are coincident.

Derivatives Bezier Curves

Although it is not necessary to numerically specify the tangent vectors at the ends of an individual Bezier curve, maintaining slope and curvature continuity when joining Bezier curves, determining surface normal for lighting or numerical control tool path calculations, or local curvature for smoothness or fairness calculations requires a knowledge of both first and second derivatives of a Bezier curve.

Recalling equation, the first derivative of a Bezier curve is

$$\mathrm{P}'(\mathrm{t}) = \sum_{i=0}^{n} B_i \, J'_{n,i}(t)$$

Second derivative is given by

$$\mathrm{P}''(\mathrm{t}) = \sum_{i=0}^{n} B_i \, J''_{n,i}(t)$$

Example on Derivatives

Consider the four points Bezier polygon

$$P(t) = B_0 J_{3,0}(t) + B_1 J_{3,1}(t) + B_2 J_{3,2}(t) + B_3 J_{3,3}(t)$$

Hence first derivative is

$$P'(t) = B_0 J'_{3,0}(t) + B_1 J'_{3,1}(t) + B_2 J'_{3,2}(t) + B_3 J'_{3,3}(t)$$

Second derivative is

$$P''(t) = B_0 J''_{3,0}(t) + B_1 J''_{3,1}(t) + B_2 J''_{3,2}(t) + B_3 J''_{3,3}(t)$$

Differentiating the basis functions directly yields

$$J_{3,0}(t) = t^0(1-t)^3 \rightarrow J'_{3,0}(t) = -3(1-t)^2 \rightarrow J''_{3,0}(t) = 6(1-t)$$

$$J_{3,1}(t) = 3t(1-t)^2 \rightarrow J'_{3,1}(t) = 3(1-t)^2 - 6t(1-t) \rightarrow J''_{3,1}(t) = -6(2-3t)$$

$$J_{3,2}(t) = 3t^2(1-t) \rightarrow J'_{3,2}(t) = 6t(1-t) - 3(t)^2 \rightarrow J''_{3,2}(t) = 6(1-3t)$$

$$J_{3,3}(t) = t^3 \rightarrow J'_{3,3}(t) = 3(t)^2 \rightarrow J''_{3,3}(t) = 6t$$

Example on Derivatives

Evaluating the results at t=0 yields

$$J'_{3,0}(0) = -3, \quad J'_{3,1}(0) = 3, \quad J'_{3,2}(0) = 0, \quad J'_{3,3}(0) = 0$$

Substituting yields

$$P'(0) = -3P_0 + 3P_1 = 3(-P_0 + P_1)$$

Thus the direction of the tangent vector at the beginning of the curve is the same as that of the first polygon span

At the end of the curve, t=1 and

$$J'_{3,0}(1) = 0, \quad J'_{3,1}(1) = 0, \quad J'_{3,2}(1) = -3, \quad J'_{3,3}(1) = 3$$

Substituting yields

$$P'(1) = -3P_2 + 3P_3 = 3(-P_2 + P_3)$$

Thus, the direction of the tangent vector at the end of the curve is the same as that of the last polygon span.

September 20, 2017 Dr. Prashant K. Jain (IIITDMJ) 96

Example on Derivatives

Evaluating the results at t=0 yields

$$J''_{3,0}(0) = 6, \quad J''_{3,1}(0) = -12, \quad J''_{3,2}(0) = 6, \quad J''_{3,3}(0) = 0$$

Substituting yields

$$P''(0) = 6B_0 - 12B_1 + 6B_2 = 6(B_0 - 2B_1 + B_2)$$

Thus second derivative or curvature of Bezier curve at start point depends on three nearest polygon points or two nearest polygon spans.

At the end of the curve, t=1 and

$$J''_{3,0}(1) = 0, \quad J''_{3,1}(1) = 6, \quad J''_{3,2}(1) = -12, \quad J''_{3,3}(1) = 6$$

Substituting yields

$$P''(1) = 6B_1 - 12B_2 + 6B_3 = 6(B_1 - 2B_2 + B_3)$$

Thus second derivative or curvature of Bezier curve at end point depends on three nearest polygon points or two nearest polygon spans.

September 20, 2017 Dr. Prashant K. Jain (IIITDMJ) 4

*****